

INTEGRALS AND POTENTIALS OF DIFFERENTIAL 1-FORMS ON THE SIERPINSKI GASKET

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ABSTRACT. We provide a definition of integral, along paths in the Sierpinski gasket K , for differential smooth 1-forms associated to the standard Dirichlet form \mathcal{E} on K . We show how this tool can be used to study the potential theory on K . In particular, we prove: i) a de Rham reconstruction of a 1-form from its periods around lacunas in K ; ii) a Hodge decomposition of 1-forms with respect to the Hilbertian energy norm; iii) the existence of potentials of smooth 1-forms on a suitable covering space of K . We finally show that this framework provides versions of the de Rham duality theorem for the fractal K .

1. INTRODUCTION

1.1. Purpose of the work. The aim of this work is to develop, on the fractal set K known as *Sierpinski gasket*, a notion of *line integral*

$$\int_{\gamma} \omega$$

along oriented paths γ in K for a class of *differential 1-forms* ω on K . The purpose for doing this is twofold: on the one hand, we wish to set up tools useful to contribute to the potential theory of K , studied in particular by Kigami [20], Strichartz [32], see also the recent work by Koskela and Zhou [23]; on the other hand, our intention is to use them to construct local representations, i.e. by integrals, of topological invariants of K .

Our approach is based on the existence of a differential calculus underlying any regular Dirichlet space X , developed in [7], [6] and further investigated in [8] and [25] for fractal spaces. There, the differential bimodule of universal 1-forms $\Omega^1(\mathcal{F})$ on the algebra of finite energy functions \mathcal{F} on X is endowed with a quadratic form Q associated with the Dirichlet energy. By separation and completion one gets a Hilbert \mathcal{F} -bimodule \mathcal{H} , called the *tangent module* of \mathcal{E} , whose elements are termed *differential 1-forms* on X , together with a derivation $\partial : \mathcal{F} \rightarrow \mathcal{H}$, i.e. a map satisfying the Leibniz rule $\partial(fg) = (\partial f)g + f(\partial g)$ $f, g \in \mathcal{F}$. Such derivation is a differential square root of the Dirichlet form in the sense that

$$\mathcal{E}[a] = \|\partial a\|_{\mathcal{H}}^2 \quad a \in \mathcal{F}.$$

Our main results are: define the integral of elements of $\Omega^1(\mathcal{F})$ along (a suitable class of) oriented paths in K , and show that this integral passes to the quotient w.r.t. Q , hence is well defined on the space $\Omega^1(K) := \Omega^1(\mathcal{F})/\{Q = 0\}$, whose elements we call smooth 1-forms; establish de Rham first and second Theorems, by proving that the sequence of periods around lacunas gives rise to a unique *harmonic form*; prove a Hodge Theorem, namely

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each cohomology class in a suitably defined de Rham cohomology group $H^1(K)$ has a unique harmonic representative; the establishment of a pairing between the cohomology of forms and the Čech homology group of the gasket (de Rham duality theorem); and finally, the construction of an (abelian) projective covering, related to the Uniform Universal Covering introduced in [3], where potentials of 1-forms will be defined.

The classical framework we refer to is that of harmonic integrals on differentiable manifolds, developed by de Rham [9] and Hodge [16]. There, the notions of differential 1-form and line integral are direct outcome of the notion of tangent bundle. The analytic tool of exterior differentiation of forms then naturally provides homotopy invariants by means of the differential complex and its associated cohomology groups. The notion of line integral on the manifold M allows to establish a local pairing first between closed 1-forms and 1-cycles, and then between the first de Rham cohomology group $H^1(M)$ and the first singular homology group $H_1(M)$. Furthermore, the choice of a Riemannian metric on M allows to introduce the notions of co-closed and harmonic forms in such a way that each cohomology class in $H^1(M)$ has a unique harmonic representative.

Trying to develop the above framework on the Sierpinski gasket K , two main problems have to be tackled.

The first is that K is not a manifold: it was originally introduced in [30] as an example of space with a dense set of ramification points so that it has no open sets homeomorphic to Euclidean domains. This is the reason why a notion of differentiable structure on K has to be introduced in an unconventional way. We choose to do so by using the notion of energy or Dirichlet form, a sort of generalized Dirichlet integral, developed by Beurling and Deny [4], that can be considered on locally compact Hausdorff spaces. In particular, we consider the so called standard Dirichlet form \mathcal{E} considered by Kusuoka [24] in his construction of a diffusion process on K and studied by Fukushima and Shima [12] and by Kigami [20] in his framework of harmonic theory on self-similar fractal sets like K . The primary role of \mathcal{E} is to provide the class of finite energy functions \mathcal{F} , which is a dense subalgebra of the algebra of continuous functions $C(K)$, and plays the role of a Sobolev space on the gasket. More importantly, there exists a canonical first order differential calculus associated to Dirichlet forms, as developed in [7]. It is represented by a closed derivation $\partial : \mathcal{F} \rightarrow \mathcal{H}$, defined on \mathcal{F} with values in a Hilbert $C(K)$ -module \mathcal{H} , which is a differential square root of the Dirichlet form in the sense that $\mathcal{E}[a] = \|\partial a\|_{\mathcal{H}}^2$. We notice that this differential structure, namely the module \mathcal{H} and the derivation ∂ , is essentially unique and only depends upon the quadratic form \mathcal{E} defined on \mathcal{F} and not on the choice of a reference measure on K . One of the main technical issues will be the proof that the integral along oriented paths makes sense on (suitably regular) elements of \mathcal{H} . As we shall see below, this will force us to a long detour: the introduction of the bimodule of universal 1-forms $\Omega^1(\mathcal{F})$ on the Dirichlet algebra \mathcal{F} , the definition of line-integrals on it, and then the proof that an element of $\Omega^1(\mathcal{F})$ with zero Hilbert norm has zero integral along all edges, namely the integral makes sense on the quotient. What we get then is an \mathcal{F} -module $\Omega^1(K)$, which densely embeds in \mathcal{H} , thus furnishing the smooth subspace on which line integrals make sense.

The second problem is that K is a topological space which is not semilocally simply connected, so that it has no universal covering, i.e. a simply connected covering space [26]. This fact affects the development of a potential theory on K . In an ordinary manifold M , any closed form ω has a pull back $\tilde{\omega}$ on the universal covering space \tilde{M} , which is obviously still closed but also exact, since \tilde{M} is simply connected. Hence, any closed form on a manifold admits a primitive function U on \tilde{M} , in the sense that $dU = \tilde{\omega}$. Moreover, the primitive U is

a potential of ω in the sense that its line integral along a path γ in M can be computed by the formula

$$\int_{\gamma} \omega = U(p) - U(q)$$

where $q, p \in \widetilde{M}$ are the initial and final points, respectively, of a lifting $\widetilde{\gamma}$ in \widetilde{M} of γ .

For the needs of a potential theory on the gasket K , the role played by the universal covering of a manifold, acted upon by its fundamental group, will be played by the Uniform Universal Cover \widetilde{K} introduced by Berestovskii and Plaut [3], and more precisely by its abelian counterpart \widetilde{L} , acted upon by the first Čech homology group $\check{H}_1(K)$, which is a projective limit of finitely generated abelian groups. In particular, the potentials U of 1-forms on K will be affine functions on \widetilde{L} .

1.2. Main results. We now come to a closer look at our results. Our first step is the definition of line integrals of the elements of the bimodule of universal 1-forms $\Omega^1(\mathcal{F})$ on the Dirichlet algebra \mathcal{F} along elementary paths in K , namely finite unions of consecutive oriented edges in K . Also, we consider a quadratic form Q on $\Omega^1(\mathcal{F})$ such that $Q[df] = \mathcal{E}[f]$, as in the tangent bimodule construction. Now we have two natural quotients to take on $\Omega^1(\mathcal{F})$, either w.r.t. the intersection of the kernels of the functionals $\omega \mapsto \int_e \omega$, where e is any edge, or w.r.t. the kernel of the quadratic form Q . A main task will be to show that the kernels coincide, hence both the integrals and Q make sense on the quotient. While the proof that Q makes sense on the space $\Omega^1(K)$ of forms modulo forms with zero integral on edges is quite direct, the converse is not at all trivial. What we do is to analyze periods of forms ω in $\Omega^1(K)$ around the lacunas of the gasket, and show that, given such periods, we may construct another form ω' with the same periods in a canonical way as a series of a suitable sequence of forms dz_{σ} , parametrized by lacunas of K . We then prove that the difference $\omega - \omega'$ between the original form and the series is an exact form dU , thus showing at once the first and second de Rham theorems for the gasket, namely the fact that one may build a form given its periods, and the fact that such a form is indeed unique, up to exact forms. In the same time, since the forms dz_{σ} are harmonic, we obtain a Hodge theorem, i.e. we show that any form has a harmonic representative in the space of (closed) forms modulo exact ones. Finally, since the decomposition of a form $\omega \in \Omega^1(K)$

$$\omega = dU + \sum_{\sigma} k_{\sigma} dz_{\sigma}$$

consists of pairwise orthogonal summands w.r.t. Q , we have that $Q[\omega] = 0$ implies $k_{\sigma} = 0$ for all σ , and $\mathcal{E}[U] = 0$, namely $\omega = 0$, thus proving that $\Omega^1(K)$ coincides with the image of $\Omega^1(\mathcal{F})$ in \mathcal{H} under the quotient map, hence is dense in the tangent module \mathcal{H} . As a further outcome of our analysis, it turns out that the only natural definition of an external differential on 1-forms giving a differential complex is the trivial one, namely all 1-forms are closed, in accordance with the fact that the gasket is topologically one-dimensional.

A second major issue of our paper is the attempt of extending the integral of a form from elementary paths to more general ones, construct potentials of 1-forms, and prove a de Rham duality theorem. The space on which potentials of 1-forms will be defined is the projective limit \widetilde{L} of a sequence of regular abelian covering spaces \widetilde{L}_n , where all loops around lacunas of order up to n are unfolded. Such pro-covering is acted upon by the Čech homology group $\check{H}_1(K, \mathbb{Z})$ of the gasket, and is an abelian counterpart of the Uniform Universal Covering

space introduced in [3]. The results just mentioned will take two different versions, a purely algebraic one and a more analytical one.

The first version concerns locally exact forms. This subspace is the natural one from the point of view of algebraic topology, first because the integral of such forms extends naturally to all curves in the gasket; second, because any locally exact form ω has a potential U_ω on \tilde{L} , such that the integral of ω along a path γ coincides with the variation of U_ω at the end-points of a lifting of γ to \tilde{L} . Moreover, the potential U_ω is associated with a homomorphism $\varphi_\omega : \check{H}_1(K, \mathbb{R}) \rightarrow \mathbb{R}$ such that $U_\omega(gx) = U_\omega(x) + \varphi_\omega(g)$. The pairing $\langle \omega, g \rangle = \varphi_\omega(g)$ extends to a de Rham duality between $\check{H}_1(K, \mathbb{R})$ and the space of locally exact forms modulo exact ones.

The second version concerns a suitable completion of closed smooth forms. Indeed, in contrast with the classical situation, the space of locally exact smooth forms is a proper subspace of the space of closed smooth forms. Enlarging the class of forms as to contain all smooth forms will correspond to a restriction of a class of allowed paths. We observe that smooth forms satisfy an estimate which puts them in a Banach space \mathcal{H}_N strictly contained in \mathcal{H} . Then we prove that forms in \mathcal{H}_N may be integrated along all paths satisfying a dual estimate. Such paths are said to have finite effective length. Such finiteness can be read on the pro-covering as well. There, we define a (possibly infinite) distance d , which splits the space \tilde{L} in d -components made of points with mutually finite distance, and selects a normal subgroup Γ_N of $\check{H}_1(K, \mathbb{Z})$. Then, paths with finite effective length are those whose lifting lies in a single d -component, and homology classes of closed paths with finite effective length are elements of Γ_N . We then prove that any form in \mathcal{H}_N has a Γ_N -affine potential on any given d -component of \tilde{L} , and the integral of a form in \mathcal{H}_N along a path γ with finite effective length coincides with the variation of the potential at the end points of a lifting of γ . Moreover, such integral gives a nondegenerate pairing between Γ_N and the space \mathcal{H}_N modulo exact forms, more precisely the latter is the Banach space dual of $\Gamma_N \otimes_{\mathbb{Z}} \mathbb{R}$.

2. THE SPACE OF 1-FORMS ON THE GASKET

2.1. Preliminary notions. We denote by K the Sierpinski gasket, one of the most studied self-similar fractal sets. It was introduced in [30] as a curve with a dense set of ramified points and has been the object of investigations in Probability [24], Theoretical Physics [28] and Mathematical Analysis [12, 20, 32].

Let $p_0 := (0, 0)$, $p_1 := (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $p_2 := (1, 0)$ be the vertices of an equilateral triangle and consider the contractions w_i of the plane: $x \in \mathbb{R}^2 \rightarrow p_i + \frac{1}{2}(x - p_i) \in \mathbb{R}^2$. Then K is the unique fixed-point w.r.t. the contraction map $E \mapsto \cup_{i=0}^2 w_i(E)$ in the set of all compact subsets of \mathbb{R}^2 , endowed with the Hausdorff metric. Two ways of approximating K are shown in Figures 1 and 2.

Let us denote by $\Sigma_m := \{0, 1, 2\}^m$ the set of words composed by m letters chosen in the alphabet $\{0, 1, 2\}$, and by $\Sigma := \bigcup_{m \geq 0} \Sigma_m$ the whole vocabulary (by definition $\Sigma_0 := \{\emptyset\}$). A word $\sigma \in \Sigma_m$ has, by definition, length m , and this is denoted by $|\sigma| := m$. For $\sigma = \sigma_1 \sigma_2 \dots \sigma_m \in \Sigma_m$, let us denote by w_σ the contraction $w_\sigma := w_{\sigma_1} \circ w_{\sigma_2} \circ \dots \circ w_{\sigma_m}$.

Let $V_0 := \{p_0, p_1, p_2\}$ be the set of vertices of the equilateral triangle and $E_0 := \{e_0, e_1, e_2\}$ the set of its edges, with e_i opposite to p_i . Then, for any $m \geq 1$, $V_m := \bigcup_{|\sigma|=m} w_\sigma(V_0)$ is the set of vertices of a finite graph (*i.e.* a one-dimensional simplex) (V_m, E_m) whose edges are given by $E_m := \bigcup_{|\sigma|=m} w_\sigma(E_0)$ (see Figure 2). The self-similar set K can be reconstructed

also as an Hausdorff limit either of the increasing sequence V_m of vertices or of the increasing sequence E_m of edges, of the above finite graphs. Set $V_* := \cup_{m=0}^{\infty} V_m$, and $E_* := \cup_{m=0}^{\infty} E_m$.

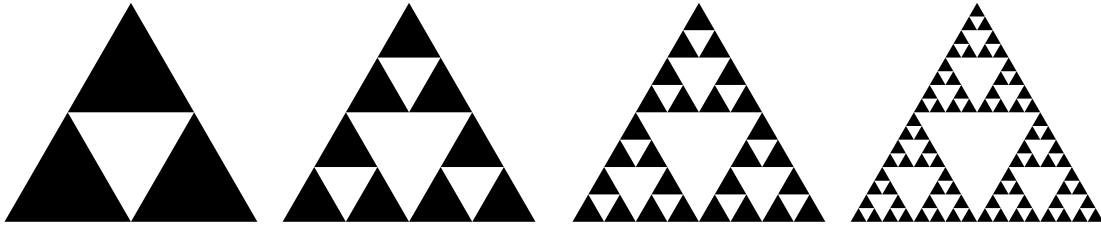


FIGURE 1. Approximations from above of the Sierpinski gasket.

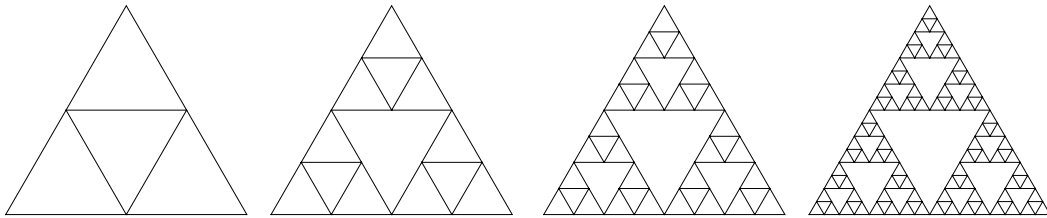


FIGURE 2. Approximations from below of the Sierpinski gasket.

In the present work a central role is played by the quadratic form $\mathcal{E} : C(K) \rightarrow [0, +\infty]$ given by

$$\mathcal{E}[f] = \lim_{m \rightarrow \infty} \left(\frac{5}{3} \right)^m \sum_{e \in E_m} |f(e_+) - f(e_-)|^2,$$

where each edge e has been arbitrarily oriented, and e_-, e_+ denote its source and target. It is a regular Dirichlet form since it is lower semicontinuous, densely defined on the subspace $\mathcal{F} := \{f \in C(K) : \mathcal{E}[f] < \infty\}$ and satisfies the *Markovianity property*

$$(2.1) \quad \mathcal{E}[f \wedge 1] \leq \mathcal{E}[f] \quad f \in C(K).$$

The existence of the limit above and the mentioned properties are consequences of the theory of *harmonic structures* on self-similar sets developed by Kigami [20]. As a result of the theory of Dirichlet forms [4, 13], the domain \mathcal{F} is an involutive subalgebra of $C(K)$ and, for any fixed $f, g \in \mathcal{F}$, the functional

$$(2.2) \quad \mathcal{F} \ni h \mapsto \frac{1}{2} (\mathcal{E}(f, gh) - \mathcal{E}(f\bar{g}, h) + \mathcal{E}(\bar{h}f, g))$$

extends to a continuous functional on $C(K)$ so that it can be represented by a finite Radon measure called the *energy measure* (or *carré du champ*) of f and g and denoted by $\mu(f, g)$. In particular, for $f \in \mathcal{F}$, $\mu(f, f)$ is a nonnegative measure and one has the representation

$$\mathcal{E}[f] = \int_K 1 d\mu(f, f) = \mu(f, f)(K) \quad f \in \mathcal{F}.$$

In applications, f may represent a configuration of a system, $\mathcal{E}[f]$ its corresponding total energy and $\mu(f, f)$ represents its energy distribution.

In the present work we will denote with $\Omega^1(\mathcal{F})$ the \mathcal{F} -bimodule of *universal 1-forms* [14] on \mathcal{F} , that is $\Omega^1(\mathcal{F})$ is the sub- \mathcal{F} -bimodule of the algebraic tensor product $\mathcal{F} \otimes \mathcal{F}$, generated by elements of the form $f dg$, where the differential operator d is defined by $df := f \otimes 1 - 1 \otimes f$,

$f \in \mathcal{F}$ and the bimodule operations are given by $f dg = f(g \otimes 1 - 1 \otimes g) := fg \otimes 1 - f \otimes g$ and $dg f := d(gf) - g df = g \otimes f - 1 \otimes gf$, $f, g \in \mathcal{F}$.

As observed in [7], in a general regular Dirichlet space over a locally compact, separable Hausdorff space X , the properties of the Dirichlet form give rise to a positive semi-definite inner product on $\Omega^1(\mathcal{F})$ given by the linear extension of the form

$$(2.3) \quad Q(fdg, hdk) = \int_X \bar{f}h d\mu(g, k) \quad f, g, h, k \in \mathcal{F}.$$

By separation and completion, this gives rise to a Hilbert space \mathcal{H} which is in fact a Hilbert $C_0(X)$ -bimodule called the *tangent bimodule associated to \mathcal{E}* and whose elements are called *square integrable forms*. In the present case of the Sierpinski gasket, since the Dirichlet form is strongly local, the right and left actions coincide so that \mathcal{H} is a Hilbert $C(K)$ -module. Moreover, the derivation $\partial : \mathcal{F} \rightarrow \mathcal{H}$, associated to the Dirichlet form \mathcal{E} comes directly from the universal derivation $d : \mathcal{F} \rightarrow \Omega^1(\mathcal{F})$ in such a way that

$$(2.4) \quad Q(fdg, hdk) = (f\partial g, h\partial k)_{\mathcal{H}}, \quad Q[df] = \|\partial f\|_{\mathcal{H}}^2 = \mathcal{E}[f] \quad f, g, h, k \in \mathcal{F}.$$

The Dirichlet or energy form \mathcal{E} should be considered as a Dirichlet integral on the gasket. It is closable with respect to any Borel regular probability measure on K which is positive on open sets and vanishes on finite sets (see [20] Theorem 3.4.6 and [22] Theorem 2.6). Once such a measure m has been chosen, \mathcal{E} gives rise to a positive, self-adjoint operator on $L^2(K, m)$, which may be thought of as a Laplace-Beltrami operator on K . However, since in the present work only the Dirichlet form plays a role, we content ourselves with the measure-valued Laplacian, as studied in [21].

A function $f \in \mathcal{F}$ is said to be *harmonic* in a open set $A \subset K$ if, for any $g \in \mathcal{F}$ vanishing on A^c , one has

$$\mathcal{E}(f, g) = 0.$$

As a consequence of the Markovianity property (2.1), a Maximum Principle holds true for harmonic functions on the gasket [20]. In particular, one calls *0-harmonic* a function u on K which is harmonic in V_0^c . Equivalently, for given boundary values on V_0 , u is the unique function in \mathcal{F} such that $\mathcal{E}[u] = \min \{\mathcal{E}[v] : v \in \mathcal{F}, v|_{V_0} = u\}$. More generally, one may call *m-harmonic* a function that, given its values on V_m , minimizes the energy among all functions in \mathcal{F} . For such functions we have

$$\mathcal{E}[u] = \left(\frac{5}{3}\right)^m \sum_{e \in E_m} |u(e_+) - u(e_-)|^2.$$

It is not difficult to check that $f \in \mathcal{F}$ is *m-harmonic* if and only if Δf is a linear combination of Dirac measures supported on the vertices V_m .

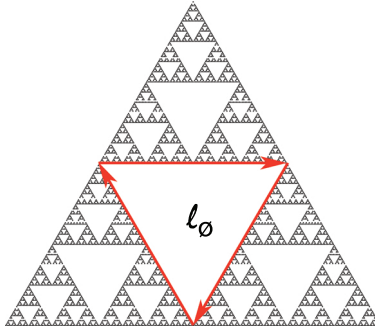
Definition 2.1. (Cells, lacunas) For any word $\sigma \in \Sigma_m$, define a corresponding *cell* in K as follows

$$C_\sigma = w_\sigma(K),$$

its *perimeter* by $\pi C_\sigma = w_\sigma(E_0)$, its (combinatorial) *boundary* by $\partial C_\sigma = w_\sigma(V_0)$ and its (combinatorial) *interior* by $C_\sigma^\circ = C_\sigma \setminus \partial C_\sigma$. We will also define the *lacuna* ℓ_\emptyset , see Fig. 3, as the topological boundary of the first removed triangle according to the approximation in Fig. 1. For any $\sigma \in \Sigma$, the lacuna ℓ_σ is defined as $\ell_\sigma = w_\sigma(\ell_\emptyset)$.

For a function f on K , let us define its *oscillation on a closed subset* $T \subseteq K$ as

$$\text{Osc}(f)(T) := \max_{x, y \in T} |f(x) - f(y)| = \max_T f - \min_T f.$$

FIGURE 3. The lacuna ℓ_\emptyset

It is easy to check that if f is a harmonic function in the interior of a cell C and C_1 is one of its three sub-cells, then $\text{Osc}(f)(C_1) \leq \frac{3}{5} \text{Osc}(f)(C)$ (see for example [32] Chapter 1 Exercise 1.3.6).

Recall that $\Omega^1(\mathcal{F})$ denotes the bimodule of universal 1-forms over the algebra \mathcal{F} . There is a natural pairing between elements of $\mathcal{F} \otimes \mathcal{F}$ and oriented edges which is given by $(f \otimes g)(e) := f(e_+)g(e_-)$ on elementary tensors. As a consequence,

$$(2.5) \quad dg(e) = g(e_+) - g(e_-)$$

$$(2.6) \quad (f dg)(e) = f(e_+)dg(e)$$

$$(2.7) \quad (dg f)(e) = f(e_-)dg(e).$$

2.2. Integrating 1-forms along elementary paths.

Definition 2.2. A path in K given by a finite union of consecutive oriented edges in E_* is called *elementary*.

Let γ be an oriented elementary path in K and $\omega = \sum_{i \in I} f_i dg_i \in \Omega^1(\mathcal{F})$. For $n \in \mathbb{N}$, define

$$I_n(\gamma)(\omega) = \sum_{e \in E_n(\gamma)} \omega(e),$$

where $E_n(\gamma)$ denotes the set of oriented edges of level n contained in γ .

Definition 2.3. We define the integral of a 1-form ω along an elementary path γ as the limit $\int_\gamma \omega = \lim_{n \rightarrow \infty} I_n(\gamma)(\omega)$.

Remark 2.4. The integral of 1-forms defined above is a kind of Riemann-Stieltjes integral conditioned to dyadic partitions of edges. Unfortunately, while the classical result of Young [36] for $\int f dg$ requires Hölder continuity of f and g with sum of the exponents > 1 , restrictions to edges of finite energy functions on the gasket are known to be only β -Hölder, with $\beta < 1/2$ (cf. e.g. [18]), therefore we cannot use Young result. Also, restrictions to edges of finite energy functions are not of bounded variation in general¹, therefore we cannot use Lebesgue-Stieltjes integral either. Nevertheless, on identifying an edge $e \in E_*$ with $[0, 1]$, the bilinear form $(f, Dg)_e$ on $L^2(e)$ given by $\int_0^1 f(x)g'(x) dx$ for f, g smooth functions, naturally extends to a bounded form on $H^{1/2}(e)$, hence makes sense also for $f, g \in \mathcal{F}$ since, by results of Jonsson [19], traces of finite energy functions on edges $e \in E_*$ belong to the fractional Sobolev space

¹In [1], p.18, examples are given of finite energy functions with non BV restriction to edges, but it is observed that harmonic functions do have BV restriction to edges. As a consequence, the integral of the form in Proposition 2.34 along an elementary path makes sense as a Lebesgue-Stieltjes integral.

$H^\alpha(e)$ for $\alpha \leq \alpha_0$, $\alpha_0 = \frac{\log(10/3)}{\log 4} \sim 0.87$. The existence of the limit in Definition 2.3 and the coincidence of the two notions are proved below.

Theorem 2.5. *Let $\omega \in \Omega^1(\mathcal{F})$ be a 1-form and γ an elementary path in K . Then*

- (i) *the integral $\int_\gamma \omega$ is well defined,*
- (ii) *the integral is a bimodule trace, namely*

$$\int_\gamma h \omega = \int_\gamma \omega h \quad h \in \mathcal{F},$$

- (iii) *for all $h \in \mathcal{F}$, the following approximation holds true:*

$$\int_\gamma h \omega = \lim_n \sum_{e \in E_n(\gamma)} h(e_+) \int_e \omega.$$

- (iv) *Let e be an edge in K , f, g finite energy functions on K . Then*

$$(2.8) \quad \int_e f dg = (f, Dg)_e.$$

Proof. It is not restrictive to assume $\omega = f dg$. We choose $n_0 \in \mathbb{N}$ such that γ is a finite union of edges of level n_0 .

- (i) For $n \geq n_0$ and $e \in E_n(\gamma)$, let $e^0 \in V_{n+1}$ be the middle point of the edge e . One computes

$$(2.9) \quad \begin{aligned} I_{n+1}(fdg) &= \sum_{e \in E_n(\gamma)} f(e_+)(g(e_+) - g(e^0)) + \sum_{e \in E_n(\gamma)} f(e^0)(g(e^0) - g(e_-)) \\ &= I_n(fdg) + \sum_{e \in E_n(\gamma)} (f(e^0) - f(e_+))(g(e^0) - g(e_-)), \end{aligned}$$

so that

$$(2.10) \quad \begin{aligned} |I_{n+1}(fdg) - I_n(fdg)| &\leq \left(\sum_{e \in E_n(\gamma)} |f(e^0) - f(e_+)|^2 \right)^{1/2} \left(\sum_{e \in E_n(\gamma)} |g(e^0) - g(e_-)|^2 \right)^{1/2} \\ &\leq \left(\sum_{e \in E_{n+1}(\gamma)} |df(e)|^2 \right)^{1/2} \left(\sum_{e \in E_{n+1}(\gamma)} |dg(e)|^2 \right)^{1/2} \end{aligned}$$

$$(2.11) \quad \leq \frac{1}{2} \sum_{e \in E_{n+1}(\gamma)} (|df(e)|^2 + |dg(e)|^2)$$

$$(2.12) \quad \leq \frac{1}{2} \left(\frac{3}{5} \right)^{n+1} (\mathcal{E}[f] + \mathcal{E}[g]).$$

Hence,

$$|I_n(\gamma)(fdg) - I_{n+p}(\gamma)(fdg)| \leq \sum_{k=n}^{n+p-1} |I_{k+1}(fdg) - I_k(fdg)| \leq \frac{3}{4}(\mathcal{E}[f] + \mathcal{E}[g]) \left(\frac{3}{5} \right)^n,$$

namely the sequence $I_n(\gamma)(fdg)$ converges.

- (ii) The result follows from

$$I_n(\gamma)(h fdg) - I_n(\gamma)(fdg h) \leq \|f\|_\infty \sum_{e \in E_n(\gamma)} |dh(e)| |dg(e)| \leq \frac{1}{2} \|f\|_\infty (\mathcal{E}[h] + \mathcal{E}[g]) \left(\frac{3}{5} \right)^n.$$

(iii) The thesis follows from

$$\begin{aligned}
\left| I_n(\gamma)(h\omega) - \sum_{e \in E_n(\gamma)} h(e_+) \int_e \omega \right| &\leq \sum_{e \in E_n(\gamma)} |h(e_+)| \left| \omega(e) - \int_e \omega \right| \\
&\leq \|h\|_\infty \sum_{e \in E_n(\gamma)} \sum_{p=0}^{\infty} |I_{p+n+1}(e)(fdg) - I_{p+n}(e)(fdg)| \\
&\leq \frac{1}{2} \|h\|_\infty \sum_{p=0}^{\infty} \sum_{e \in E_n(\gamma)} \sum_{e' \in E_{p+n+1}(e)} (|df(e')|^2 + |dg(e')|^2) \\
&\leq \frac{1}{2} \|h\|_\infty \sum_{p=0}^{\infty} \sum_{e' \in E_{p+n+1}(\gamma)} (|df(e')|^2 + |dg(e')|^2) \\
&\leq \frac{1}{2} \|h\|_\infty (\mathcal{E}[f] + \mathcal{E}[g]) \sum_{p=0}^{\infty} \left(\frac{3}{5} \right)^{p+n+1} \leq \frac{3}{4} \|h\|_\infty (\mathcal{E}[f] + \mathcal{E}[g]) \left(\frac{3}{5} \right)^n.
\end{aligned}$$

(iv) Given a function f on an edge e , consider the continuous piecewise-linear approximation f_n which coincides with f on diadic points of e identified with the interval $[0, 1]$:

$$f_n(x) = \sum_{j=1}^{2^n} \chi_{[(j-1)2^{-n}, j2^{-n})}(x) \left(f((j-1)2^{-n}) + \frac{f(j2^{-n}) - f((j-1)2^{-n})}{2^{-n}} (x - (j-1)2^{-n}) \right).$$

Since eq. (2.8) clearly holds for continuous piecewise-linear functions, it is sufficient to show that both terms in (2.8) are continuous w.r.t. the approximation above. By definition, $I_k(fdg) = I_k(f_n dg_n)$, $n \geq k$, therefore

$$\left| \int_e fdg - \int_e f_n dg_n \right| \leq \left| \int_e fdg - I_n(fdg) \right| + \left| I_n(f_n dg_n) - \int_e f_n dg_n \right| \rightarrow 0,$$

since the first summand goes to 0 by the preceding Theorem 2.5, and, setting $|e| = p$,

$$\left| I_n(f_n dg_n) - \int_e f_n dg_n \right| = \sum_{e' \in E_{p+n}(e)} df(e') dg(e') \leq \frac{1}{2} \left(\frac{3}{5} \right)^{n+p} (\mathcal{E}[f] + \mathcal{E}[g]).$$

As for the bilinear form, it is sufficient to show that $f_n \rightarrow f$ in $H^{1/2}(e)$. According to [19], a norm for the Sobolev spaces $H^\alpha[0, 1]$, $1/2 < \alpha < 1$, is

$$\|f\|_{H^\alpha} = (f(0)^2 + f(1)^2)^{1/2} + \left(\sum_{n=0}^{\infty} 2^{n(2\alpha-1)} E_n(f) \right)^{1/2},$$

where

$$E_n(f) = \sum_{j=1}^{2^n} (f(j2^{-n}) - f((j-1)2^{-n}))^2.$$

Therefore,

$$\|f - f_k\|_{H^\alpha}^2 = \sum_{n=k+1}^{\infty} 2^{n(2\alpha-1)} E_n(f - f_k) \leq 2 \sum_{n=k+1}^{\infty} 2^{n(2\alpha-1)} E_n(f) + 2 \sum_{n=k+1}^{\infty} 2^{n(2\alpha-1)} E_n(f_k).$$

If $\alpha \leq \alpha_0$, the first summand is a remainder of a convergent series, hence goes to 0, as $k \rightarrow \infty$. As for the second, since f_k has constant slope on diadic intervals of length 2^{-k} , a

direct computation shows that, for $n > k$, $E_n(f_k) = 2^{k-n}E_k(f)$, herefore

$$\sum_{n=k+1}^{\infty} 2^{n(2\alpha-1)} E_n(f_k) = (2^{2-2\alpha} - 1)^{-1} 2^{k(2\alpha-1)} E_k(f) \rightarrow 0$$

since $2^{k(2\alpha-1)} E_k(f)$ is the generic term of a convergent series. This shows that, for $\alpha \in (1/2, \alpha_0]$, $f_k \rightarrow f$ in $H^\alpha([0, 1])$. The convergence in $H^{1/2}[0, 1]$ then follows. \square

Our aim now is to show that the integral defined above on $\Omega^1(\mathcal{F})$ makes sense on (sufficiently regular) elements of the tangent module, namely to show that the integral passes to the quotient w.r.t. forms in the kernel of the quadratic form Q described in eq. (2.3). However, in order to achieve this result, we have to take a different quotient, namely to identify forms whose integrals coincide on any edge, and to dwell in such space for a while. After proving a series of results, which have an interest in their own, we will be able to prove that the latter quotient indeed coincides with the former, i.e.

$$(2.13) \quad \int_e \omega = 0 \quad \forall e \in E_* \iff Q[\omega] = 0.$$

Definition 2.6. Let us now introduce the equivalence relation on $\Omega^1(\mathcal{F})$ given by $\omega \sim \omega' \iff \int_e(\omega - \omega') = 0$, for all $e \in E_*$, and consider the quotient space $\Omega^1(K) := \Omega^1(\mathcal{F}) / \sim$. We call *smooth 1-forms* the elements of $\Omega^1(K)$.

In the following, we use the shorthand notation $\mathcal{E}_C[f] := \mathcal{E}[f|_C]$, for any cell C in K .

Lemma 2.7. For any $\omega \in \Omega^1(\mathcal{F})$,

$$(2.14) \quad Q[\omega] = \lim_{n \rightarrow \infty} (5/3)^n \sum_{e \in E_n} \left| \int_e \omega \right|^2.$$

As a consequence, the quadratic form Q is well defined on the space $\Omega^1(K)$.

Proof. Let us set

$$(2.15) \quad Q_n[\omega] = (5/3)^n \sum_{e \in E_n} \left| \int_e \omega \right|^2, \quad \tilde{Q}_n[\omega] = (5/3)^n \sum_{e \in E_n} |\omega(e)|^2, \quad \omega \in \Omega^1(\mathcal{F}).$$

We have

$$\tilde{Q}_n[f dg - dg f] = \left(\frac{5}{3}\right)^n \sum_{e \in E_n} df(e)^2 dg(e)^2 \leq \mathcal{E}_n[f] \max_{e \in E_n} dg(e)^2,$$

hence $\lim_n \tilde{Q}_n[f dg - dg f] = 0$. A straightforward computation gives

$$\tilde{Q}_n(dg, f dh) + \tilde{Q}_n(dg, dh f) = \mathcal{E}_n(g, fh) - \mathcal{E}_n(gh, f) + \mathcal{E}_n(h, fg),$$

therefore

$$\begin{aligned} \tilde{Q}_n(dg, f dh) &= \frac{1}{2} \left(\tilde{Q}_n(dg, f dh) + \tilde{Q}_n(dg, dh f) + \tilde{Q}_n(dg, f dh - dh f) \right) \\ &= \frac{1}{2} (\mathcal{E}_n(g, fh) - \mathcal{E}_n(gh, f) + \mathcal{E}_n(h, fg)) + \frac{1}{2} \tilde{Q}_n(dg, f dh - dh f) \\ &\rightarrow \frac{1}{2} (\mathcal{E}(g, fh) - \mathcal{E}(gh, f) + \mathcal{E}(h, fg)) = \int_K f d\mu(g, h), \end{aligned}$$

therefore $\tilde{Q}_n \rightarrow Q$. We finally prove that the two limits $\lim_{n \rightarrow \infty} Q_n[\omega]$, $\lim_{n \rightarrow \infty} \tilde{Q}_n[\omega]$ coincide. For sequences $x = \{x_e : e \in E_*\}$, we introduce the seminorms

$$(2.16) \quad \Phi_n(x) := \left(\frac{5}{3}\right)^{n/2} \left(\sum_{e \in E_n} |x_e|^2\right)^{1/2}.$$

In particular, $\tilde{Q}_n[\omega] = \Phi_n(\omega(e))^2$ and $Q_n[\omega] = \Phi_n(\int_e \omega)^2$. Let us denote with $C(e)$ the cell having e as one of its boundary segments. We get, by inequality (2.10),

$$\begin{aligned} \Phi_n \left((f_i dg_i)(e) - \int_e f_i dg_i \right)^2 &= \left(\frac{5}{3}\right)^n \sum_{e \in E_n} \left| I_n(e)(f_i dg_i) - \lim_{k \rightarrow \infty} I_k(e)(f_i dg_i) \right|^2 \\ &\leq \left(\frac{5}{3}\right)^n \sum_{e \in E_n} \left(\sum_{j=n}^{\infty} |I_{j+1}(e)(f_i dg_i) - I_j(e)(f_i dg_i)| \right)^2 \\ &\leq \left(\frac{5}{3}\right)^n \sum_{e \in E_n} \left(\sum_{j=n}^{\infty} \left(\frac{3}{5}\right)^{j+1} \mathcal{E}_{C(e)}[f_i]^{1/2} \mathcal{E}_{C(e)}[g_i]^{1/2} \right)^2 \\ &= \frac{9}{4} \left(\frac{3}{5}\right)^n \sum_{e \in E_n} \mathcal{E}_{C(e)}[f_i] \mathcal{E}_{C(e)}[g_i] \leq \frac{27}{4} \left(\frac{3}{5}\right)^n \mathcal{E}[f_i] \mathcal{E}[g_i]. \end{aligned}$$

As a consequence, for $\omega = \sum_{i \in I} f_i dg_i$,

$$\begin{aligned} |\tilde{Q}_n[\omega]^{1/2} - Q_n[\omega]^{1/2}| &= \left| \Phi_n(\omega(e)) - \Phi_n\left(\int_e \omega\right) \right| \leq \left| \Phi_n(\omega(e) - \int_e \omega) \right| \\ &\leq \sum_{i \in I} \left| \Phi_n((f_i dg_i)(e) - \int_e f_i dg_i) \right| \leq \frac{3\sqrt{3}}{2} \left(\frac{3}{5}\right)^{n/2} \sum_{i \in I} \mathcal{E}[f_i]^{1/2} \mathcal{E}[g_i]^{1/2}. \end{aligned}$$

□

The following Proposition summarizes the previous results.

Proposition 2.8. *The space $\Omega^1(K)$ is an \mathcal{F} -module and the universal derivation gives rise to a derivation $d : \mathcal{F} \rightarrow \Omega^1(K)$ (still indicated by the same symbol). The integral along an elementary path and the seminorm $Q^{1/2}$ are well defined on $\Omega^1(K)$.*

2.3. Locally exact 1-forms. On a smooth manifold M , a closed form ω is locally exact, namely $\forall x \in M$, there exists a pair (\mathcal{V}_x, f_x) , where \mathcal{V}_x is a neighborhood of x and f_x is a local potential of ω on \mathcal{V}_x , that is f_x satisfies

$$(2.17) \quad \int_{\gamma} \omega = f_x(\gamma(1)) - f_x(\gamma(0)), \quad \forall \gamma \subset \mathcal{V}_x.$$

A family of local potentials as above may be abstractly described as a pair $(\{U_i\}, \{f_i\})$ where $\{U_i\}$ is an open cover of M and f_i is a continuous function on U_i such that $(f_i - f_j)|_{U_i \cap U_j}$ is locally constant. Clearly such pairs can be considered for any topological space².

We say that $(\{U_i\}, \{f_i\})$ is equivalent to $(\{V_i\}, \{g_i\})$ if $(f_i - g_j)|_{U_i \cap V_j}$ is locally constant, denote the quotient space by $\Omega_{\text{loc}}^1 C(K)$ and call its elements *locally exact topological 1-forms*.

²On a smooth manifold, codimension-1 smooth foliations are in 1:1 correspondence with closed 1-forms, the longitudinal tangent of the former being locally described as the kernel of the 1-form, or, equivalently, as the level sets of the local potentials of the 1-form. The latter description extends to topological spaces, giving rise to codimension-1 C^0 -foliations and coincides with the pairs $(\{U_i\}, \{f_i\})$ considered above [5].

As shown below, the integral in (2.17) extends to a pairing between locally exact topological 1-forms and continuous paths in X .

Lemma 2.9. *Let X be a topological space, $(\{U_i\}_{i \in I}, \{f_i\}_{i \in I})$ a representative of a locally exact topological 1-form ω as above, and $\gamma : [0, 1] \rightarrow X$ a continuous path. Then $\int_\gamma \omega$ is well defined.*

Proof. The family $\{\gamma^{-1}(U_i) : i \in I\}$ is an open cover of $[0, 1]$, so we can consider its Lebesgue number $\delta > 0$. Let $\{t_0 = 0, t_1, \dots, t_n = 1\}$ be a partition of $[0, 1]$ such that $t_k - t_{k-1} < \delta$, $k = 1, \dots, n$, so that, for any k , $\gamma([t_{k-1}, t_k]) \subset U_{i_k}$ for some $i_k \in I$. Then we define $\int_\gamma \omega := \sum_{k=1}^n f_{i_k}(\gamma(t_k)) - f_{i_k}(\gamma(t_{k-1}))$. Suppose now that $(\{V_i\}, \{g_i\})$ is another representative of ω with Lebesgue number $\delta' > 0$, $\{t'_0 = 0, t'_1, \dots, t'_m = 1\}$ the corresponding partition of $[0, 1]$ such that $t'_j - t'_{j-1} < \delta'$, $j = 1, \dots, m$, and denote by $\{s_0 = 0, s_1, \dots, s_\ell = 1\}$ the union of the two partitions. Clearly the two integrals coincide, proving the statement. \square

We now show that, when f_i 's have finite energy, any $(\{U_i\}, \{f_i\})$ gives rise to a unique element of $\Omega^1(K)$. Such elements will be called locally exact smooth 1-forms, the corresponding space will be denoted by $\Omega_{\text{loc}}^1(K)$. We first note that, because of the topology of K , any $(\{U_i\}, \{f_i\})$ may be equivalently represented by $\{f_\sigma\}_{|\sigma|=n}$ for some n , where f_σ is a local potential on the closed cell C_σ .

Proposition 2.10. *Let $\{f_\sigma\}_{|\sigma|=n}$ be a family of local potentials as above representing a locally exact 1-form ω_0 , with $\mathcal{E}_{C_\sigma}[f_\sigma] < \infty$ for any σ . Then there exists a unique 1-form $\omega \in \Omega^1(K)$ such that $\int_\gamma \omega = \int_\gamma \omega_0$ for any elementary path γ . Such ω will be called n -exact. Moreover, $Q[\omega] = \sum_{|\sigma|=k} \mathcal{E}_{C_\sigma}[f_\sigma]$ and $Q^{1/2}$ is a norm on the space $\Omega_{\text{loc}}^1(K)$ of locally exact forms with finite-energy local potentials.*

Proof. We may associate with any f_σ in the statement an element in $\Omega^1(K)$ as follows: let $A \supset C_\sigma$ be an open set in K such that the connected components of $(K \setminus C_\sigma^\circ) \cap \bar{A}$ are cells containing exactly one boundary vertex of C_σ ; let \tilde{f}_σ be a function in \mathcal{F} which coincides with f_σ in C_σ and is constant on each connected component of $(K \setminus C_\sigma^\circ) \cap \bar{A}$; and let χ_σ be a function in \mathcal{F} which is 1 on C_σ and has support contained in A . If we set $\omega_\sigma = \chi_\sigma d\tilde{f}_\sigma$, then

$$\int_e \omega_\sigma = \lim_{n \rightarrow \infty} \sum_{\substack{e' \in E_n \\ e' \subset e}} \chi_\sigma(e'_+) (\tilde{f}_\sigma(e'_+) - \tilde{f}_\sigma(e'_-)).$$

Now, if e intersects C_σ at most in one vertex, we get $\int_e \omega_\sigma = 0$, because \tilde{f}_σ is constant on any $e' \in E_n$, $e' \subset e$. If, on the contrary, $e \subset C_\sigma$, then $\chi_\sigma(e'_+) = 1$, for any such e' , while $\tilde{f}_\sigma = f_\sigma$, so that $\int_e \omega_\sigma = \lim_{n \rightarrow \infty} \sum_{\substack{e' \in E_n \\ e' \subset e}} (f_\sigma(e'_+) - f_\sigma(e'_-)) = f_\sigma(e_+) - f_\sigma(e_-)$. Clearly $\sum_{|\sigma|=n} \omega_\sigma$ is the required n -exact form. We now prove the second statement. For any $n > k$, we get

$$Q_n(\omega) = \left(\frac{5}{3}\right)^n \sum_{e \in E_n} \left| \int_e \omega \right|^2 = \left(\frac{5}{3}\right)^n \sum_{|\tau|=k} \sum_{e \in E_n(C_\tau)} \left| \int_e \omega \right|^2 = \sum_{|\tau|=k} \mathcal{E}_n[f_\tau].$$

Therefore, $Q(\omega) = \lim_{n \rightarrow \infty} Q_n(\omega) = \sum_{|\sigma|=k} \lim_{n \rightarrow \infty} \mathcal{E}_n[f_\sigma] = \sum_{|\sigma|=k} \mathcal{E}[f_\sigma]$. Finally, $0 = Q(\omega) = \sum_{|\sigma|=k} \mathcal{E}[f_\sigma] \implies f_\sigma$ is constant on C_σ , for any $\sigma \implies \omega = 0$. \square

We now introduce a distinguished system of locally exact smooth 1-forms associated with lacunas which will play a fundamental role in the following.

Definition 2.11. For any $n \geq 0$ and $|\sigma| = n$, define dz_σ as the $(n+1)$ -exact form which minimizes the norm $Q^{1/2}$ among those $(n+1)$ -exact 1-forms ω satisfying $\int_{\ell_\sigma} \omega = 1$.

By definition, dz_σ is exact in any of the cells $C_{\sigma i}$, hence $\int_{\pi C_\sigma} dz_\sigma = -1$ (lacunas are traversed clockwise and perimeters of cells are traversed counter-clockwise, according to the standard convention, as they constitute the boundary of the union of the convex hulls of the cells $C_{\sigma i}$, $i = 1, 2, 3$). The minimization request implies that dz_σ vanishes in any cell C_τ with $\tau \neq \sigma$, $|\tau| = n$, and that dz_σ is symmetric for rotations of $\frac{2}{3}\pi$ around ℓ_σ .

Proposition 2.12. (i) *The forms dz_σ are weakly co-closed, i.e. orthogonal to all exact smooth 1-forms, and pairwise orthogonal, with*

$$(2.18) \quad Q[dz_\sigma] = \frac{5}{6} \left(\frac{5}{3} \right)^{|\sigma|}.$$

(ii) *Any n -exact topological form has a unique decomposition as the sum of an exact topological form plus a finite linear combination of dz_τ , $|\tau| < n$. If the form is smooth, the decomposition is orthogonal w.r.t. the quadratic form Q . Uniqueness of the decomposition implies that $Q^{1/2}$ is a norm on locally exact smooth forms.*

Proof. (i) A simple calculation shows that for any cell $C_{\sigma i}$, the local potential z_σ^i on such cell is the harmonic function determined (up to an additive constant) by the values $\frac{1}{6}, 0, -\frac{1}{6}$ on the vertices x_1, x_2, x_3 , where x_3, x_1 is the edge bounding the lacuna. Therefore, Δz_σ^i may be canonically identified with the measure given by the linear combination $\frac{1}{2}\delta_{x_1} - \frac{1}{2}\delta_{x_3}$. As a consequence, for any $f \in \mathcal{F}$,

$$Q(df, dz_\sigma) = \sum_{i=1,2,3} Q(df, dz_\sigma^i) = \sum_{i=1,2,3} \mathcal{E}(f, z_\sigma^i) = \sum_{i=1,2,3} \int_K f d(\Delta z_\sigma^i) = 0.$$

If $\tau < \sigma$ the orthogonality follows as above; if τ and σ are not ordered, dz_σ and dz_τ have disjoint support. The value of the norm follows from a direct computation.

(ii) We note that, for any cell C_σ , an $(n+1)$ -exact form ω on C_σ is indeed n -exact if and only if $\int_{\ell_\sigma} \omega = 0$, since in this case the three local potentials on the three sub-cells may glue to a continuous function on C_σ . Therefore, any $(n+1)$ -exact form ω supported in C_σ may be written as

$$\omega = (\omega - c_\sigma dz_\sigma) + c_\sigma dz_\sigma, \quad c_\sigma := \int_{\ell_\sigma} \omega,$$

namely, for any cell C_σ , the codimension of n -exact forms into $(n+1)$ -exact forms supported in C_σ is 1. This shows that exact forms and the dz_τ , $|\tau| < n$, generate the n -exact forms, hence the thesis. When the form is smooth, the exact part in the decomposition is also smooth, hence the statement follows by Proposition 2.12. \square

Similarly to the case of an ordinary smooth manifold, 1-forms which are locally exact and co-closed will be termed *harmonic*, therefore $\{dz_\sigma : \sigma \in \Sigma\}$ is an orthogonal system of harmonic 1-forms. A more general result is contained in Lemma 2.26.

2.4. Winding numbers and a combinatoric way to describe lacunas bounding cells.

Since dz_σ is invariant under rotations of $\frac{2}{3}\pi$ around the lacuna ℓ_σ , the integral along any edge e bounding C_σ is equal to $-1/3$. We now consider the integral $B_{\rho\tau} = \int_{\ell_\tau} dz_\rho$. It is not difficult

to see that $B_{\rho\tau}$ does not vanish only if $\tau \leq \rho$ (τ is a truncation of ρ), more precisely,

$$B_{\rho\tau} = \int_{\ell_\tau} dz_\rho = \begin{cases} 1 & \text{if } \tau = \rho, \\ -1/3 & \text{if } \ell_\tau \cap \pi C_\rho \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $B = \{B_{\rho\tau}\}$ is a lower unitriangular matrix with indices in Σ (i.e. $B_{\rho\tau} = 0$ for $\tau > \rho$ and $B_{\rho\rho} = 1$ for any ρ). The following result is well known for finite matrices and for infinite matrices with indices in \mathbb{Z} [17], but extends to infinite matrices with indices in a partially ordered set Σ such that $\{\tau \in \Sigma : \tau \leq \sigma\}$ is finite and linearly ordered, cf. also [10] for the case of finitary matrices.

Proposition 2.13. *The set $UT(\Sigma, \mathbb{R})$ of \mathbb{R} -valued lower unitriangular matrices with indices in Σ is a group contained in $Aut(\mathbb{R}^\Sigma)$.*

Let us observe that the product and the action on \mathbb{R}^Σ are defined in a purely algebraic sense, since the sums involved are always finite. Setting $A_{\sigma\rho} = (B^{-1})_{\sigma\rho}$, we get

$$(2.19) \quad \int_{\ell_\tau} \sum_{\rho \leq \sigma} A_{\sigma\rho} dz_\rho = \delta_{\sigma\tau} \quad \sigma, \tau \in \Sigma.$$

Remark 2.14 (Winding number). In other words, the 1-form $\omega^\sigma := \sum_{\rho \leq \sigma} A_{\sigma\rho} dz_\rho$ detects only the lacuna ℓ_σ . As a consequence, for any closed path γ in K ,

$$\int_\gamma \omega^\sigma$$

is the *winding number of the path γ around the lacuna ℓ_σ* .

Lemma 2.15. *With the notation above, $0 \leq A_{\sigma\tau} \leq 1$, $\tau \leq \sigma$.*

Proof. Since $A = B^{-1}$, we have $A^* = (1 - B^*)A^* + 1$, hence, setting $D = 3(1 - B^*)$, we get

$$(2.20) \quad A_{\sigma\tau} = \frac{1}{3} \sum_{\tau \leq \rho \leq \sigma} D_{\tau\rho} A_{\sigma\rho} + \delta_{\sigma\tau}.$$

For a given σ , let us rename indices and variables as follows: replace the n -th truncation $\sigma^{(n)}$ of σ with n , so that the order is reversed, and rename $A_{\sigma\sigma^{(n)}}$ as v_n . Then the equation above becomes

$$v_p = \frac{1}{3} \sum_{j=0}^p D_{pj} v_j + \delta_{0p}, \quad p = 0, 1, \dots, |\sigma|.$$

Denoting by P the projection on the 0-th component, we get $v = (\frac{1}{3}D + P)v$. Recall that D_{ij} may be non zero for at most three indices i following j , and observe that D is a lower triangular matrix, hence $(D^p)_{jk}$ does not vanish only if $k \leq j - p$, and $PD = 0$. Therefore we get

$$v = \left(\frac{1}{3}D + P\right)^p v = 3^{-p} D^p v + \sum_{j=0}^{p-1} \left(\frac{1}{3}\right)^j D^j P v,$$

and, since $v_0 = 1$,

$$v_p = 3^{-p} (D^p)_{p0} v_0 + \sum_{j=0}^{p-1} \left(\frac{1}{3}\right)^j (D^j)_{p0} v_0 = \sum_{j=0}^p \left(\frac{1}{3}\right)^j (D^j)_{p0}.$$

Since, by definition, the entries of D are either 0 or 1, we may interpret D as the adjacency matrix of an oriented simple graph, where the vertices are the indices $0, 1, \dots, |\sigma|$ and an oriented edge goes from j to i if $D_{ij} = 1$. Then, $(D^j)_{p0}$ is equal to the number of oriented paths of length j joining 0 with p . Since from any vertex may depart at most three edges, if there is an edge joining 0 with p , then there are at most 2 oriented paths of length 2 joining 0 with p , at most 6 oriented paths of length 3 joining 0 with p , and so on. So, denoting with n_i the number of oriented paths of length i joining 0 with p , we have

$$(2.21) \quad \begin{cases} n_1 \leq 1 \\ n_1 + n_2 \leq 3 \\ 3n_1 + n_2 + n_3 \leq 9 \\ \dots \\ \sum_{i=1}^{q-1} 3^{q-1-i} n_i + n_q \leq 3^{q-1}. \end{cases}$$

As a consequence, for $q \geq 1$, we have

$$v_q = \sum_{i=1}^q 3^{-i} n_i = 3^{-q} n_q + 3^{1-q} \left(\sum_{i=1}^{q-1} 3^{q-1-i} n_i \right) \leq 3^{-q} n_q + 3^{1-q} (3^{q-1} - n_q) \leq 1 - \frac{2}{3} 3^{1-q} n_q \leq 1.$$

□

2.5. $\Omega^1(K)$ embeds in the tangent module. We introduce here a completion of $\Omega^1(K)$ w.r.t. a given norm. This completion (and norm) will play only an auxiliary role, being used in the proof that $\Omega^1(K)$ can be equivalently defined as the quotient of $\Omega^1(\mathcal{F})$ w.r.t. the \mathcal{H} -norm. But, this completion has some pathologies, cf. Proposition 2.25, in particular does not embed in \mathcal{H} , therefore such norm will be abandoned later on. By making use of the quadratic forms Q_n defined in (2.15), we endow $\Omega^1(K)$ with the norm

$$(2.22) \quad \|\omega\|_{\sup} = \sup_n Q_n[\omega]^{1/2}.$$

Since $Q_n \rightarrow Q$ on $\Omega^1(K)$, $\|\omega\|_{\sup}$ is finite on it. Since $\|\omega\|_{\sup} = 0 \Rightarrow Q_n[\omega] = 0, \forall n \Rightarrow \int_e \omega = 0, \forall e \in E_*$, the norm property follows. Let us observe that the integrals $\omega \rightarrow \int_e \omega$ and the seminorm $Q^{1/2}$ are continuous w.r.t. the norm $\|\cdot\|_{\sup}$. We denote by $\overline{\Omega^1(K)}^{\sup}$ the completion of $(\Omega^1(K), \|\cdot\|_{\sup})$. Clearly, the quadratic forms $Q, Q_n, n \in \mathbb{N}$, extend to $\overline{\Omega^1(K)}^{\sup}$ by continuity.

Let us now consider the space $\ell_N(\Sigma) := \{a \in \mathbb{R}^{\Sigma} : N(a) < \infty\}$, where the functional N is given by $N(a) = \sup_{n \geq 0} (5/3)^n \sum_{|\sigma|=n} |a_{\sigma}|$. Clearly, N is a norm on such a space.

Lemma 2.16. *Upper triangular matrices on Σ with bounded entries belong to $B(\ell_N(\Sigma))$.*

Proof. Let T be such a matrix, $v \in \ell_N(\Sigma)$, so that $\sum_{|\tau|=n} |(Tv)_{\tau}| \leq \sum_{|\tau|=n} \sum_{\sigma \geq \tau} |T_{\tau\sigma}| \cdot |v_{\sigma}| \leq \|T\|_{\infty} \sum_{|\sigma| \geq n} |v_{\sigma}| \leq \|T\|_{\infty} N(v_{\bullet}) \sum_{k \geq n} (3/5)^k = \frac{5}{2} \|T\|_{\infty} N(v_{\bullet}) (3/5)^n$, where $\|T\|_{\infty} = \sup_{\sigma\tau} |T_{\sigma\tau}|$, implying that $N(Tv) \leq \frac{5}{2} \|T\|_{\infty} N(v_{\bullet})$. □

Lemma 2.17. *The sequence $\{c_{\sigma} := \int_{\ell_{\sigma}} \omega\}$ of periods of a smooth 1-form ω belongs to $\ell_N(\Sigma)$.*

Proof. It is enough to prove the result for $\omega = fdg$. Observe that

$$|c_{\sigma}| = \left| \lim_n I_n(\ell_{\sigma})(fdg) \right| \leq |I_{|\sigma|+1}(\ell_{\sigma})(fdg)| + \sum_{k=|\sigma|+1}^{\infty} |I_{k+1}(\ell_{\sigma})(fdg) - I_k(\ell_{\sigma})(fdg)|.$$

Since ℓ_σ is a closed curve, $|I_{|\sigma|+1}(\ell_\sigma)(fdg)| = |I_{|\sigma|+1}(\ell_\sigma)((f - \text{const})dg)|$. Denoting by x_1, x_2, x_3 the vertices of ℓ_σ , and choosing $\text{const} = f(x_1)$, we get

$$|I_{|\sigma|+1}(\ell_\sigma)(fdg)| = |df(x_1, x_2)dg(x_1, x_2) + df(x_1, x_3)dg(x_2, x_3)| \leq \frac{1}{2} \sum_{e \in E_{|\sigma|+1}(\ell_\sigma)} df(e)^2 + dg(e)^2.$$

By (2.11) we get $|c_\sigma| \leq \frac{1}{2} \sum_{k=|\sigma|+1}^{\infty} \sum_{e \in E_k(\ell_\sigma)} (df(e)^2 + dg(e)^2)$, whence $\sum_{|\sigma|=n} |c_\sigma| \leq \frac{5}{4} \left(\frac{3}{5}\right)^{n+1} (\mathcal{E}[f] + \mathcal{E}[g])$. The thesis follows. \square

Lemma 2.18. *If $c = \{c_\sigma\}$ belongs to $\ell_N(\Sigma)$, then $k := A^*c \in \ell_N(\Sigma)$.*

Proof. Immediate by Lemmas 2.15 and 2.16. \square

Proposition 2.19. *Let $c = \{c_\sigma\}$ belong to $\ell_N(\Sigma)$, and set $k = A^*c$. Then, the series $\sum_\sigma k_\sigma dz_\sigma$ converges to a form $\omega_H \in \overline{\Omega^1(K)}^{\text{sup}}$, having the c_σ 's as its periods. In particular, if $\omega \in \Omega^1(K)$, $c_\sigma := \int_{\ell_\sigma} \omega$, k and ω_H as above, then ω and ω_H have the same periods.*

Proof. A simple calculation shows that $Q_n[dz_\sigma] \leq (5/3)^{|\sigma|}$, therefore $\|k_\sigma dz_\sigma\|_{\text{sup}}^2 = \sup_n Q_n[k_\sigma dz_\sigma] \leq |k_\sigma|^2 (5/3)^{|\sigma|}$. Then the series converges absolutely in $\overline{\Omega^1(K)}^{\text{sup}}$, since, by Lemma 2.18,

$$\sum_\sigma \|k_\sigma dz_\sigma\|_{\text{sup}} \leq \sum_\sigma (5/3)^{|\sigma|/2} |k_\sigma| \leq N(k_\bullet) \sum_k (3/5)^{k/2} = \left(1 - \sqrt{3/5}\right)^{-1} N(k_\bullet).$$

In particular, $\int_{\ell_\tau} \omega_H = \sum_\sigma k_\sigma \int_{\ell_\tau} dz_\sigma$. By the results in Section 2.4, $AB = BA = 1$, $|A_{\sigma\tau}| \leq 1$ and $|B_{\sigma\tau}| = |\int_{\ell_\tau} dz_\sigma| \leq 1$, hence $A^*, B^* \in B(\ell_N(\Sigma))$, by Lemma 2.16. Then,

$$\int_{\ell_\tau} \omega_H = \sum_\sigma B_{\sigma\tau} \sum_{\rho \geq \sigma} A_{\rho\sigma} c_\rho = (B^* A^* c)_\tau = ((AB)^* c)_\tau = c_\tau.$$

\square

Lemma 2.20. *Let ω be a smooth 1-form. Then, for any σ ,*

$$\int_{\pi C_\sigma} \omega = - \sum_{\tau \geq \sigma} \int_{\ell_\tau} \omega = - \sum_{\tau \geq \sigma} \int_{\ell_\tau} \omega_H = \int_{\pi C_\sigma} \omega_H.$$

Proof. As above, we may assume $\omega = fdg$. As for the first equation, we have, for any $n \geq |\sigma|$,

$$\int_{\pi C_\sigma} fdg = - \sum_{\tau \geq \sigma, |\tau| \leq n} \int_{\ell_\tau} fdg + \sum_{\tau \geq \sigma, |\tau| = n+1} \int_{\pi C_\tau} fdg.$$

Therefore we have to prove that the second summand goes to 0 when $n \rightarrow \infty$. It is not restrictive to assume $\sigma = \emptyset$. With estimates similar to those in Lemma 2.17, we get

$$\begin{aligned} \sum_{|\tau|=n+1} \int_{\pi C_\tau} fdg &\leq \frac{1}{2} \sum_{|\tau|=n+1} \sum_{k=n+1}^{\infty} \sum_{e \in E_k(\pi C_\tau)} df(e)^2 + dg(e)^2 \\ &\leq \frac{1}{2} \sum_{k=n+1}^{\infty} \left(\frac{3}{5}\right)^k (\mathcal{E}[f] + \mathcal{E}[g]) \leq \frac{3}{4} \left(\frac{3}{5}\right)^n (\mathcal{E}[f] + \mathcal{E}[g]). \end{aligned}$$

The second equation follows by Proposition 2.19, and the third by absolute convergence. \square

Lemma 2.21. *Let $\omega, k_\sigma, \omega_H$ be as above, and let γ be an elementary simple path contained in the cell C_σ , $|\sigma| = n$. Then,*

$$\left| \int_\gamma \omega_H \right| \leq N(k_\bullet)(n+3) \left(\frac{3}{5} \right)^n.$$

Proof. It is easy to see that $\int_\gamma dz_\tau$ can be non-zero only if either $\tau < \sigma$ or $\tau \geq \sigma$. Moreover, since γ has no loops, $|\int_\gamma dz_\tau| \leq 1$.

When $\tau < \sigma$, choosing i such that $\tau i \leq \sigma$, dz_τ is exact in $C_{\tau i}$, hence in C_σ , with $\text{Osc}_{C_{\tau i}}(z_\tau) = 1/3$. Since the oscillation of a harmonic function in a sub-cell is bounded by $3/5$ times the oscillation of the original cell (cf. e.g. ex. 1.3.6 p. 8 in [32]), we get

$$(2.23) \quad \left| \int_\gamma dz_\tau \right| \leq \text{Osc}_{C_\sigma}(z_\tau) \leq \frac{1}{3} \left(\frac{3}{5} \right)^{|\sigma| - |\tau| - 1}.$$

Since, by Lemma 2.18, $\{k_\sigma\} \in \ell_N(\Sigma)$,

$$\begin{aligned} \left| \int_\gamma \omega_H \right| &\leq \sum_\tau |k_\tau| \left| \int_\gamma dz_\tau \right| \leq \sum_{\tau \geq \sigma} |k_\tau| + \frac{1}{3} \sum_{\tau < \sigma} |k_\tau| \left(\frac{3}{5} \right)^{n - |\tau| - 1} \leq \sum_{|\tau| \geq n} |k_\tau| + \frac{1}{3} \sum_{|\tau| < n} |k_\tau| \left(\frac{3}{5} \right)^{n - |\tau| - 1} \\ &\leq N(k_\bullet) \sum_{j \geq n} \left(\frac{3}{5} \right)^j + \frac{1}{3} N(k_\bullet) \sum_{j < n} \left(\frac{3}{5} \right)^{n-1} \leq N(k_\bullet)(n+3) \left(\frac{3}{5} \right)^n. \end{aligned}$$

□

Let us now consider the form $\omega_1 = \omega - \omega_H$, which has trivial integral along the perimeter of any cell C_σ . For any n , denoting by S_n the 1-skeleton of the n -th approximation of K , given two points $x, y \in S_n$, and a path γ in S_n joining them, the integral $\int_\gamma (\omega - \omega_H)$ depends only on the end points x, y , namely we get a primitive function U_E^n on S_n , i.e.,

$$(2.24) \quad \forall e \in E_n, \quad \int_e (\omega - \omega_H) = dU_E^n(e).$$

Lemma 2.22. *Let $\omega = fdg$, ω_H and U_E^n be as above. Set $|\sigma| = n$, and choose $x_0 \in V_n \cap C_\sigma$, $x \in V_{n+p} \cap C_\sigma$. Then there exists a constant c such that*

$$|U_E^{n+p}(x) - U_E^{n+p}(x_0)| \leq \|f\|_\infty \text{Osc}_{C_\sigma}(g) + c(\mathcal{E}[f] + \mathcal{E}[g])(n+3)(3/5)^n.$$

Proof. First step. Let $\sigma^0, \sigma^1, \dots, \sigma^p$ be the subsequent multi-indices of length $n+j$, $\sigma^0 = \sigma$, such that $x \in C_{\sigma^j}$, $j = 0, \dots, p$. We shall construct inductively a path γ , joining x_0 with x , given by vertices $x_0, \dots, x_{p+1} = x$, such that

- $x_j \in V_{n+j}$ for $j \leq p$, $x_{p+1} \in V_{n+p}$;
- $x_j \in C_{\sigma^j}$, $j \leq p$;
- either $x_{j-1} = x_j$, or x_{j-1}, x_j are joined by an edge e_j , with $e_j \in E_{n+j}$ if $0 < j \leq p$, and $e_{p+1} \in E_{n+p}$. In the first case we set e_j to be the trivial edge.

Since x_0 is given, we only need to describe the inductive step. Suppose we have x_{j-1} , $j \leq p$. If $x_{j-1} \in C_{\sigma^j}$, we set $x_j := x_{j-1}$. If not, it is connected by an edge $e_j \in E_{n+j}$ to a vertex $x_j \in V_{n+j} \cap C_{\sigma^j}$. Finally, x_p and x_{p+1} are both vertices in $V_{n+p} \cap C_{\sigma^p}$, hence either coincide or are joined by an edge e_{p+1} .

Second step. There exists a constant c_1 such that

$$\left| \int_\gamma fdg \right| \leq \|f\|_\infty \text{Osc}_{C_\sigma}(g) + c_1 \left(\frac{3}{5} \right)^n (\mathcal{E}[f] + \mathcal{E}[g]).$$

We decompose the restriction of f to γ as $f = \sum_{k=0}^{p+1} f_k$, with $f_0 = f(x_0)$ constantly, and, for $0 < k \leq p+1$,

$$f_k(t) = \begin{cases} 0 & t \in e_j, j < k, \\ f(t) - f(x_{k-1}) & t \in e_k, \\ f(x_k) - f(x_{k-1}) & t \in e_j, j > k. \end{cases}$$

We then get

$$\begin{aligned} \int_{\gamma} f dg &= \int_{\gamma} f_0 dg + \sum_{k=1}^{p+1} \sum_{j=k}^{p+1} \int_{e_j} f_k dg \\ &= f(x_0)(g(x) - g(x_0)) + \sum_{k=1}^{p+1} \int_{e_k} f_k dg + \sum_{k=1}^p \sum_{j=k+1}^{p+1} df(e_k) dg(e_j) \end{aligned}$$

As for the first summand, we clearly have $|f(x_0)(g(x) - g(x_0))| \leq \|f\|_{\infty} \text{Osc}_{C_{\sigma}}(g)$. We now estimate the second summand. First observe that

$$\begin{aligned} \left| \int_{e_k} f_k dg \right| &\leq \left(I_{n+k}(e_k)(f_k dg) + \sum_{r=n+k+1}^{\infty} |I_r(e_k)(f dg) - I_{r-1}(e_k)(f dg)| \right) \\ &\leq \sum_{r=n+k}^{\infty} \left(\sum_{e \in E_r} df(e)^2 \right)^{1/2} \left(\sum_{e \in E_r} dg(e)^2 \right)^{1/2} \leq \frac{5}{4} \left(\frac{3}{5} \right)^{n+k} (\mathcal{E}[f] + \mathcal{E}[g]). \end{aligned}$$

Therefore,

$$\left| \sum_{k=1}^{p+1} \int_{e_k} f_k dg \right| \leq \sum_{k=1}^{p+1} \frac{5}{4} \left(\frac{3}{5} \right)^{n+k} (\mathcal{E}[f] + \mathcal{E}[g]) \leq \frac{15}{8} \left(\frac{3}{5} \right)^n (\mathcal{E}[f] + \mathcal{E}[g]).$$

We now consider the third summand. Since, $\forall e \in E_m$, $|df(e)| \leq (3/5)^{m/2} \mathcal{E}[f]^{1/2}$, we get

$$\begin{aligned} \left| \sum_{k=1}^p \sum_{j=k+1}^{p+1} df(e_k) dg(e_j) \right| &\leq \mathcal{E}[f]^{1/2} \mathcal{E}[g]^{1/2} \sum_{k=1}^{\infty} \left(\frac{3}{5} \right)^{(n+k)/2} \sum_{j=k+1}^{\infty} \left(\frac{3}{5} \right)^{(n+j)/2} \\ &= \frac{3}{4} \frac{\sqrt{3}}{\sqrt{5} - \sqrt{3}} \left(\frac{3}{5} \right)^n (\mathcal{E}[f] + \mathcal{E}[g]). \end{aligned}$$

The thesis follows.

Conclusion. Since $|U_E^{n+p}(x) - U_E^{n+p}(x_0)| = |\int_{\gamma}(f dg - \omega_H)| \leq |\int_{\gamma} f dg| + |\int_{\gamma} \omega_H|$, the result follows by Step 2 and Lemma 2.21. \square

Proposition 2.23. *For any $\omega \in \Omega^1(K)$, there exists $U_E \in \mathcal{F}$ and $\omega_H \in \Omega^1(K)$ such that $\omega = dU_E + \omega_H$, where $\omega_H = \sum_{\sigma} k_{\sigma} dz_{\sigma}$.*

Proof. As usual, it is not restrictive to assume $\omega = f dg$. Clearly, the functions U_E^n constructed above are defined up to an additive constant, therefore we choose a vertex x in V_0 and set $U_E^n(x) = 0$ for any n . Let us now observe that the functions U_E^n satisfy, for $m \geq n$, $U_E^m|_{S_n} = U_E^n$, therefore they define a function U_E on $S := \cup_n S_n$. By Lemma 2.22, U_E is uniformly continuous on a dense subset of K , hence it extends to a continuous function on

K , and, by definition, $\int_e (\omega - \omega_H) = dU_E(e)$. This shows that

$$Q_n[\omega - dU_E - \sum_{|\sigma| \leq k} k_\sigma dz_\sigma] = Q_n[\sum_{|\sigma| > k} k_\sigma dz_\sigma],$$

therefore, reasoning as in the proof of Proposition 2.19,

$$\|\omega - dU_E - \sum_{|\sigma| \leq k} k_\sigma dz_\sigma\|_{\sup} \leq \sup_n \sum_{|\sigma| > k} Q_n[k_\sigma dz_\sigma]^{1/2} = N(k_\bullet) \left(1 - \sqrt{3/5}\right)^{-1} \left(\frac{3}{5}\right)^{(k+1)/2} \rightarrow 0.$$

This shows that $\omega - \omega_H = dU_E$ as elements of $\overline{\Omega^1(K)}^{\sup}$ and $\mathcal{E}[U_E] = Q[\omega - \omega_H] < \infty$. \square

Theorem 2.24. *For $\omega \in \Omega^1(K)$, $\int_e \omega = 0 \ \forall e \in E_*$ iff $Q[\omega] = 0$, i.e. $Q^{1/2}$ is a norm in $\Omega^1(K)$. Therefore, $\Omega^1(K)$ coincides with $\Omega^1(\mathcal{F})/\{Q = 0\} \subset \mathcal{H}$, $Q^{1/2}$ coincides with $\|\cdot\|_{\mathcal{H}}$, and $\overline{\Omega^1(K)}^{\mathcal{H}} = \mathcal{H}$.*

Proof. Since, by Proposition 2.12 (i), the decomposition $\omega = dU_E + \sum_{\sigma} k_\sigma dz_\sigma$ is an orthogonal decomposition w.r.t. the norm $Q^{1/2}$, then $Q[\omega] = 0$ implies $Q[dU_E] = \mathcal{E}[U_E] = 0$ and $k_\sigma = 0$ for any $\sigma \in \Sigma$. As a consequence ω vanishes. The last statement follows by the definition of \mathcal{H} . \square

Let us recall that $Q^{1/2} = \lim_n Q_n^{1/2}$ while $\|\cdot\|_{\sup} = \sup_n Q_n^{1/2}$. Since the second norm is stronger than the first, the first extends by continuity to a functional on the completion $\overline{\Omega^1(K)}^{\sup}$ of $\Omega^1(K)$ w.r.t. the norm $\|\cdot\|_{\sup}$. Contrary to the case of Sobolev spaces, where completions w.r.t. stronger norms imbeds into those with weaker norms, here Cauchy sequences which are equivalent w.r.t. the weaker norm are not so w.r.t. the stronger, so that $Q^{1/2}$ is only a seminorm on the completion $\overline{\Omega^1(K)}^{\sup}$ of $\Omega^1(K)$ w.r.t. the norm $\|\cdot\|_{\sup}$.

Proposition 2.25. *$Q^{1/2}$ is not a norm on $\overline{\Omega^1(K)}^{\sup}$.*

Proof. We illustrate the strategy of the proof. First, we describe a sequence ω_n in $\Omega^1(K)$, then we construct a normed space A in which $(\Omega^1(K), \|\cdot\|_{\sup})$ isometrically embeds, and a unit vector $\omega \in A$ such that $\|\omega - \omega_n\|_{\sup} \rightarrow 0$, showing that the sequence ω_n is Cauchy and $\lim_n \|\omega_n\|_{\sup} = 1$. Finally, we observe that $\lim_n Q[\omega_n] = 0$.

Let $p_i, i = 0, 1, 2$, be the external vertices of the gasket, e_i be the edge in E_0 opposite to p_i , $i = 0, 1, 2$, and let g be the 0-harmonic function taking value $-1/2$ on x_0 , 0 on x_1 and $1/2$ on x_2 . Then, for any given n , let us consider the n -exact form ω_n determined by the functions $f_\sigma, |\sigma| = n$, where

$$(2.25) \quad f_\sigma = \begin{cases} 2^{-n} g \circ (w_\sigma)^{-1} & \text{if } \sigma \in \{0, 2\}^n \\ 0 & \text{otherwise.} \end{cases}$$

Observe that, for any edge $e \in E_k$,

$$(2.26) \quad \lim_n \int_e \omega_n = \begin{cases} 2^{-k} & \text{if } e = w_\sigma e_1, \sigma \in \{0, 2\}^k \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the vector space $\text{Vec}(E)$ given by finite linear combinations of edges, its algebraic dual $\text{Vec}(E)^*$, where the duality is denoted by the integral, $\langle \omega, e \rangle = \int_e \omega$, and the subspace $A = \{\omega \in \text{Vec}(E)^* : \|\omega\|_{\sup} < \infty\}$, with $\|\omega\|_{\sup}^2 = \sup_n (5/3)^n \sum_{e \in E_n} |\int_e \omega|^2$. Clearly $\|\cdot\|_{\sup}$ is a norm on A , $(A, \|\cdot\|_{\sup})$ is a normed vector space, and $\Omega^1(K) \subset A$ in an obvious way. We now prove that ω_n converges to a non-trivial element $\omega \in A$, thus showing that ω_n is a Cauchy sequence in $\Omega^1(K)$ having a non-trivial limit $\omega \in \overline{\Omega^1(K)}^{\sup}$.

Define $\omega \in \text{Vec}(E)^*$ by $\int_e \omega := \begin{cases} 2^{-k} & \text{if } e = w_\sigma e_1, \sigma \in \{0, 2\}^k \\ 0 & \text{otherwise.} \end{cases}$. Since $Q_k[\omega] = (5/6)^k$, ω is a

unit element in A . We now compute $Q_k[\omega - \omega_n]$ in A .

If $k < n$,

$$Q_k[\omega - \omega_n] = \left(\frac{5}{3}\right)^k \sum_{\substack{e \in E_k \\ e \not\subset e_1}} \left| \int_e \omega_n \right|^2 = \left(\frac{5}{3}\right)^k \cdot 2^{k+1} (2^{-n-1})^2 = \frac{1}{2} \left(\frac{10}{3}\right)^k 4^{-n} < \frac{1}{2} \left(\frac{5}{6}\right)^n.$$

If $k \geq n$, we use the estimate $Q_k[\omega - \omega_n]^{1/2} \leq Q_k[\omega]^{1/2} + Q_k[\omega_n]^{1/2}$. Since each edge $e \in E_k$ is contained in only one cell C_σ , where ω_n has a potential f_σ , we have

$$Q_k[\omega_n] = \left(\frac{5}{3}\right)^k \sum_{|\sigma|=n} \sum_{e \in E_k(C_\sigma)} \left| \int_e \omega_n \right|^2 = \sum_{\sigma \in \{0,2\}^n} \mathcal{E}_{C_\sigma}[2^{-n} g \circ w_\sigma^{-1}] = \frac{3}{2} \left(\frac{5}{6}\right)^n.$$

In particular, $Q_k[\omega_n] = Q[\omega_n]$. Therefore, when $k \geq n$,

$$Q_k[\omega - \omega_n] \leq 2Q_k[\omega] + 2Q_k[\omega_n] \leq 2 \left(\frac{5}{6}\right)^k + 3 \left(\frac{5}{6}\right)^n = 5 \left(\frac{5}{6}\right)^n.$$

Hence, $\|\omega - \omega_n\|_{\sup}^2 = \sup_k Q_k[\omega - \omega_n] \leq 5 \left(\frac{5}{6}\right)^n$, namely ω_n converges in $\|\cdot\|_{\sup}$ to the non-trivial 1-form $\omega \in \overline{\Omega^1(K)}^{\sup}$. On the other hand, since Q is continuous w.r.t. the norm $\|\cdot\|_{\sup}$, $Q[\omega] = \lim_n Q[\omega_n] = \lim_n \frac{3}{2} \left(\frac{5}{6}\right)^n = 0$. \square

2.6. Hodge and De Rham Theorems. Corollary 2.24 shows that $\Omega^1(K)$ can be equivalently defined as the quotient of $\Omega^1(\mathcal{F})$ w.r.t. the quadratic form Q , hence is a dense \mathcal{F} -submodule of \mathcal{H} . Then $\Omega^1(K)$ may be considered as the space of smooth 1-forms, on which the integral along elementary paths is naturally defined. The following Lemma is the analytic counterpart of the fact that the gasket is topologically 1-dimensional.

Lemma 2.26. *Any local exterior differential on $\Omega^1(K)$ which is a closable operator on \mathcal{H} vanishes, hence co-closed forms are harmonic.*

Proof. A closable operator $(d_1, \Omega^1(K))$ on \mathcal{H} , with values in another non degenerate, Hilbertian \mathcal{F} -module $\Omega^2(K)$, and giving rise to a complex $0 \rightarrow \mathcal{F} \rightarrow \Omega^1(K) \rightarrow \Omega^2(K)$, necessarily vanishes on locally exact smooth 1-forms. However a form is locally exact iff the k_σ 's are eventually zero, as shown in Proposition 2.12 (ii), therefore they are dense in \mathcal{H} . The result follows. \square

Theorem 2.27 (Hodge decomposition). *Any 1-form $\omega \in \mathcal{H}$ can be uniquely decomposed as an orthogonal sum $dU_E \oplus \omega_H$ of an exact form and a harmonic form. The dz_σ 's give an orthogonal basis for the space of harmonic forms, therefore the decomposition above may be written as $\omega = dU_E + \sum_\sigma k_\sigma dz_\sigma$.*

Proof. We observe that the space $B^1(K)$ of exact (smooth) forms is norm closed. This has been argued in [8], and we show it here for the sake of completeness. Indeed, $B^1(K)$ is the range $d(\mathcal{F})$ of the derivation $d : \mathcal{F} \rightarrow \mathcal{H}$. Since the space of 0-harmonic functions on K is three dimensional, it is enough to prove that the image $d(\mathcal{F}_0)$ of the subspace $\mathcal{F}_0 := \{f \in \mathcal{F} : f \text{ vanishes on } V_0\}$ of finite energy functions vanishing at the boundary V_0 of K , is closed in \mathcal{H} . By the inequality $\|u\|_\infty \leq c\sqrt{\mathcal{E}[u]}$ $u \in \mathcal{F}_0$ (holding for a finite constant $c > 0$, see [20] Chapter 2), if $\{u_n \in \mathcal{F}_0 : n \geq 1\}$ is a sequence such that $\{du_n \in \mathcal{H} : n \geq 1\}$ has the Cauchy property, then $\{u_n \in \mathcal{F}_0 : n \geq 1\}$ is itself a Cauchy sequence in \mathcal{F}_0 with respect to

the uniform norm and we may consider its limit $u \in \mathcal{F}_0$. As the quadratic form \mathcal{E} comes from a harmonic structure on K (see [20] Example 3.1.5), it is the pointwise monotone limit of bounded quadratic forms on $C(K)$ and, in particular, it is lower semicontinuous. Then, if for a fixed $\varepsilon > 0$, $N \geq 1$ is such that $\mathcal{E}[u_n - u_m] < \varepsilon$, for all $n, m \geq N$, then

$$\|du - du_m\|_{\mathcal{H}}^2 = \mathcal{E}[u - u_m] \leq \liminf_n \mathcal{E}[u_n - u_m] < \varepsilon \quad m \geq N$$

so that the sequence $\{du_n \in \mathcal{H} : n \geq 1\}$ converges to $du \in \mathcal{H}$.

Finally, the space $B^1(K)^\perp$ consists of co-closed forms, which are also closed by Lemma 2.26. The result follows. \square

Remark 2.28. (1) An equivalent way to formulate Hodge decomposition theorem is that each cohomology class has a (unique) harmonic representative.

(2) Hodge decomposition allows us to define a gradient d^* on forms:

$$d^*\omega = d^*(dU_E + \omega_H) = \Delta U_E.$$

Observe that the domain and the range of d^* depend on the corresponding data for Δ .

(3) Even though the dz_σ 's are parametrized by lacunas, they are not the dual basis of the lacunas, considered as a basis for the homology vector space, as follows by eq. (2.19).

In order to formulate the first and second theorems by de Rham we need to introduce a stronger norm on \mathcal{H} such that the integral on elementary paths still makes sense on the closure of $\Omega^1(K)$ w.r.t. such norm. With N as in Section 2.5, we set $\|\omega\|_N = \mathcal{E}[U_E]^{1/2} + N(k_\bullet)$ on the square integrable forms $\omega = dU_E + \sum_\sigma k_\sigma dz_\sigma$ for which this expression is finite. We write $\mathcal{H}_N := \{\omega \in \mathcal{H} : \|\omega\|_N < \infty\}$, and note that $\Omega^1(K) \subset \mathcal{H}_N$.

Lemma 2.29. *If $\{k_\sigma\} \in \ell_N(\Sigma)$, then $\sum_\sigma k_\sigma dz_\sigma \in \mathcal{H}$.*

Proof. Eq. (2.18) gives

$$\left\| \sum_\sigma k_\sigma dz_\sigma \right\|_{\mathcal{H}}^2 = \sum_\sigma |k_\sigma|^2 \|dz_\sigma\|_{\mathcal{H}}^2 \leq \frac{5}{6} \sum_{n \geq 0} \left(\frac{5}{3}\right)^n \left(\sum_{|\sigma|=n} |k_\sigma| \right)^2 \leq \frac{25}{12} N(k_\bullet)^2.$$

\square

Let us consider the norm N' on sequences, dual to the norm N : $N'(a) = \sum_{n \geq 0} (3/5)^n \sup_{|\sigma|=n} |a_\sigma|$.

We shall say that a path $\gamma \subset K$ has *finite effective length* $\lambda(\gamma)$ if

$$(2.27) \quad \lambda(\gamma) := N'\left(\int_\gamma dz_\bullet\right) < \infty.$$

Lemma 2.30. *Edges have finite effective length. Indeed, $\lambda(e) \leq (3/5)^{n-1}(3+2n)/6$ if $e \in E_n$.*

Proof. Let $e \in E_n$, and let C_σ , $|\sigma| = n$, be the cell having e as a boundary edge. Then $\int_e dz_\tau$ is non-zero only if either $\tau \geq \sigma$ or $\tau < \sigma$. It is easy to see that, for $\tau \geq \sigma$, $\int_e dz_\tau \leq 1/3$, such value being attained e.g. for $\tau = \sigma$. For $\tau < \sigma$, one can estimate $\int_e dz_\tau$ with the oscillation of z_τ on C_σ , which in turn is estimated by $(3/5)^{|\sigma|-|\tau|-1} \text{Osc}_{C_\rho}(z_\tau)$, where C_ρ is the cell of level $|\tau| + 1$ containing C_σ . Since $\text{Osc}_{C_\rho}(z_\tau) = 1/3$, we have $\int_e dz_\tau \leq 1/3 \cdot (3/5)^{n-|\tau|-1}$. This value is not necessarily attained, but it does e.g. when e is one of the boundary edges for ℓ_τ , for the index τ immediately preceding σ . Therefore,

$$N'\left(\int_e dz_\bullet\right) = \sum_{k \geq 0} \left(\frac{3}{5}\right)^k \sup_{|\sigma|=k} \left| \int_e dz_\sigma \right| \leq \sum_{k < n} \frac{1}{3} \left(\frac{3}{5}\right)^k \left(\frac{3}{5}\right)^{n-k-1} + \sum_{k \geq n} \frac{1}{3} \left(\frac{3}{5}\right)^k = \frac{3+2n}{6} \left(\frac{3}{5}\right)^{n-1}$$

□

Theorem 2.31. *The integral extends by continuity to any $\omega = dU_E + \sum_{\sigma} k_{\sigma} dz_{\sigma} \in \mathcal{H}_N$ and any path γ with finite effective length, as*

$$(2.28) \quad \int_{\gamma} \omega = \int_{\gamma} dU_E + \sum_{\sigma} k_{\sigma} \int_{\gamma} dz_{\sigma}.$$

In particular, $k_{\tau} = \sum_{\sigma \geq \tau} A_{\sigma\tau} \int_{\ell_{\sigma}} \omega$, hence, for $\omega \in \mathcal{H}_N$, the decomposition of Theorem 2.27 is determined by the periods of ω . Therefore:

[de Rham first theorem] If $\{c_{\sigma}\} \in \ell_N(\Sigma)$, $\exists \omega_H$ harmonic in \mathcal{H}_N such that $\int_{\ell_{\sigma}} \omega_H = c_{\sigma}$.

[de Rham second theorem] If $\omega \in \mathcal{H}_N$ and $\int_{\ell_{\sigma}} \omega = 0$ for all σ , then ω is exact.

Proof. Since $\sum_{\sigma} |k_{\sigma}| \cdot |\int_{\gamma} dz_{\sigma}| \leq N(k_{\bullet}) N'(\int_{\gamma} dz_{\bullet})$ the series $\sum_{\sigma} k_{\sigma} \int_{\gamma} dz_{\sigma}$ converges. By Proposition 2.23, eq. (2.28) extends the integral of smooth forms on elementary paths. The last statement follows as in the proof of Proposition 2.19.

De Rham first theorem: if $k = A^*c$, then $\{k_{\sigma}\} \in \ell_N(\Sigma)$, by Lemma 2.16, so we get the thesis by setting $\omega_H = \sum_{\sigma} k_{\sigma} dz_{\sigma}$.

De Rham second theorem is immediate, by $k_{\tau} = \sum_{\sigma \geq \tau} A_{\sigma\tau} \int_{\ell_{\sigma}} \omega$. □

2.7. On the existence of non-locally exact forms. On a manifold, all closed forms are locally exact, namely the difference between closed and exact forms cannot be detected locally. Due to its exotic topology, this is no longer true on the gasket, as we show below.

Lemma 2.32. *Let f_i be the 0-harmonic function on the gasket taking value 1 on the vertex p_i and 0 on the others, and consider the scalar products $a_{ijk} := Q(df_i, f_j df_k)$, $i, j, k = 0, 1, 2$. Then*

$$a_{ijk} = \begin{cases} 1 & \text{if } i = j = k; \\ -\frac{1}{2} & \text{if } i = j \neq k \text{ or } i \neq j = k; \\ \frac{1}{2} & \text{if } i = k \neq j; \\ 0 & \text{if the indices are pairwise different.} \end{cases}$$

Proof. The result directly follows from the definition of Q and eq. (2.2), together with the relation

$$2Q(df_i, f_j df_j) = Q(df_i, d(f_j^2)) = \langle \Delta f_i, f_j^2 \rangle = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i \neq j; \end{cases}$$

where we recall that Δf_i is the sum of twice the Dirac measure concentrated on the vertex p_i minus the Dirac measures concentrated on the other vertices. □

Lemma 2.33. *With the notation of the previous Lemma,*

$$Q(dz_{\emptyset}, f_0 df_1) = \frac{1}{15}.$$

Proof. Since dz_{\emptyset} is invariant under $2\pi/3$ rotations, we have $Q(dz_{\emptyset}, f_0 df_1) = Q(dz_{\emptyset}, f_i df_{i+1})$ for any $i = 0, 1, 2$, hence

$$\begin{aligned} Q(dz_{\emptyset}, f_0 df_1) &= \frac{1}{3} \sum_{i=0,1,2} Q(dz_{\emptyset}, f_i df_{i+1}) = \frac{5}{9} \sum_{i,j=0,1,2} Q(dz_{\emptyset} \circ w_j, (f_i df_{i+1}) \circ w_j) \\ &= \frac{5}{3} \sum_{i=0,1,2} Q(dz_{\emptyset} \circ w_1, (f_i df_{i+1}) \circ w_1), \end{aligned}$$

where, in the last equality, we used the fact that $\sum_{i=0,1,2} Q(dz_\emptyset \circ w_j, (f_i df_{i+1}) \circ w_j)$ does not depend on j . A simple computation shows that $dz_\emptyset \circ w_1 = dg$, with $g = \frac{1}{6}(-f_0 + f_2)$, $f_0 \circ w_1 = \frac{1}{5}(2f_0 + f_2)$, $f_1 \circ w_1 = \frac{1}{5}(2 + 3f_1)$, $f_2 \circ w_1 = \frac{1}{5}(f_0 + 2f_2)$. As a consequence,

$$\begin{aligned} Q(dz_\emptyset, f_0 df_1) &= \frac{1}{15} Q(dg, 2df_0 + 4df_2 + 3f_1 df_0 + 6f_1 df_2 + 6f_0 df_1 + 3f_2 df_1 + \\ &\quad + 2f_0 df_0 + 2f_2 df_2 + f_0 df_2 + 4f_2 df_0) \\ &= \frac{1}{15} Q(dg, 2df_2 + 3f_1 df_2 - 3f_2 df_1 + 3f_2 df_0) \\ &= \frac{2}{15} \langle \Delta f_2, g \rangle + \frac{1}{30} Q(d(-f_0 + f_2), f_1 df_2 - f_2 df_1 + f_2 df_0) \end{aligned}$$

where in the second equality we used the invariance of the scalar product under the reflection of the gasket which fixes p_1 . By Lemma 2.32 the second summand vanishes, while $\langle \Delta f_2, g \rangle = 1/2$, proving the thesis. \square

Proposition 2.34. *The form $f_0 df_1$ is not locally exact, indeed all the coefficients k_σ of the decomposition of Theorem 2.27 are non-zero.*

Proof. Set $\alpha(g, h) = Q(dz_\emptyset, gdh)$. Since dz_\emptyset is harmonic,

$$\alpha(g, h) = Q(dz_\emptyset, gdh) = Q(dz_\emptyset, d(gh)) - Q(dz_\emptyset, hdg) = -Q(dz_\emptyset, hdg) = -\alpha(h, g).$$

Restricting this bilinear form to 0-harmonic functions, we get a bilinear antisymmetric form on \mathbb{R}^3 such that $\alpha(g, \text{const}) = 0$ for any g . Moreover it is non-trivial since, by Lemma 2.33, $\alpha(f_0, f_1) = 1/15$. As a consequence, $\alpha(g, h) = 0$ iff $ag + bh = 1$, for some constants a, b . For any index σ we get

$$Q(dz_\sigma, f_0 df_1) = \left(\frac{5}{3}\right)^{|\sigma|} Q(dz_\emptyset, f_0 \circ w_\sigma d(f_1 \circ w_\sigma)) = \left(\frac{5}{3}\right)^{|\sigma|} \alpha(f_0 \circ w_\sigma, f_1 \circ w_\sigma).$$

By harmonicity of f_i , the map $f_i \rightarrow f_i \circ w_\sigma$ is injective and linear, therefore $\alpha(f_0, f_1) \neq 0 \Leftrightarrow f_0$ and f_1 do not generate constants $\Leftrightarrow f_0 \circ w_\sigma$ and $f_1 \circ w_\sigma$ do not generate constants $\Leftrightarrow \alpha(f_0 \circ w_\sigma, f_1 \circ w_\sigma) \neq 0$. Finally, by Theorem 2.27, we have

$$Q(dz_\sigma, f_0 df_1) = k_\sigma Q[dz_\sigma],$$

namely $k_\sigma \neq 0$ for any σ . \square

3. POTENTIALS OF SMOOTH 1-FORMS

The first aim of this section is to prove a de Rham duality Theorem for locally exact forms, namely when only finite linear combinations of exact forms and dz_σ 's are considered. In this “algebraic” case, the integral is defined for any path in K , and locally exact forms are in one to one correspondence with suitable affine potentials (up to additive constants) on a suitable pro-covering space.

3.1. Uniform coverings of the Sierpinski gasket. Berestovskii and Plaut introduced a notion of Uniform Universal Cover for a suitable family of uniform spaces (such spaces are called coverable). As explained in [27], in the case of a connected metric space X , the inverse system giving rise to the Uniform Universal Cover \tilde{X} consists in a tower $\{X_\varepsilon\}_{\varepsilon>0}$ of regular coverings, where ε corresponds to the size of the cycles that are unfolded in the corresponding covering. When the space is also geodesic, the equivalence class of the covering actually changes only for a discrete set in $(0, \infty)$. For the gasket K of side 1 endowed with the geodesic metric induced by the embedding in \mathbb{R}^2 , the sequence is $\{\varepsilon_n = \text{const} \cdot 2^{-n}\}$, as

can be easily deduced from the content of Section 7 of [3]. The projective limit of the groups $\text{deck}(\tilde{K}_n)$, which is denoted by $\delta_1(K)$, and called the deck group of K in [3], coincides with the Čech homotopy group $\check{\pi}_1(K)$ of K , cf. [2], Proposition 2.8.

We now describe the coverings $\tilde{K}_n = K_{\varepsilon_n}$, $n \in \mathbb{N}$, of the gasket K . Let us recall that, for any n , K can be written as $K = \bigcup_{|\sigma|=n} w_\sigma(K)$. If T is the convex hull of K in the plane, and $T_n = \bigcup_{|\sigma|=n} w_\sigma(T)$, \tilde{K}_n may be seen as the regular covering of K induced by the universal covering \tilde{T}_n of T_n via the embedding $\iota_n : K \hookrightarrow T_n$. Due to the simple connectedness of \tilde{T}_n , the local potentials f_σ , $|\sigma| = n$, of a (not necessarily smooth) n -exact form ω glue together to form a continuous potential f_ω of $\tilde{\omega}$ on \tilde{K}_n . If $\tilde{\omega}$ is the $\text{deck}(\tilde{K}_n)$ -periodic form obtained by lifting ω to \tilde{K}_n , we clearly have

$$(3.1) \quad \int_\gamma \omega = \int_{\tilde{\gamma}} \tilde{\omega} = f_\omega(\tilde{\gamma}(1)) - f_\omega(\tilde{\gamma}(0)).$$

Definition 3.1. (Affine functions) Let G be a topological group acting on a space X . A continuous function f on X is G -affine if there exists a continuous group homomorphism $\varphi : G \rightarrow (\mathbb{R}, +)$ such that $f(gx) = f(x) + \varphi(g)$ for all $(g, x) \in G \times X$.

Let us observe that, since the group homomorphisms φ associated to affine functions are valued in the abelian group $(\mathbb{R}, +)$, they vanish on commutators. In particular, let $[\text{deck}(\tilde{K}_n), \text{deck}(\tilde{K}_n)]$ and $\Gamma_n := \text{Ab}(\text{deck}(\tilde{K}_n)) = \text{deck}(\tilde{K}_n)/[\text{deck}(\tilde{K}_n), \text{deck}(\tilde{K}_n)]$ be the commutator subgroup and the abelianization, respectively, of the group $\text{deck}(\tilde{K}_n)$. Then, a $\text{deck}(\tilde{K}_n)$ -affine function on \tilde{K}_n can be considered as a Γ_n -affine function on the quotient space

$$\tilde{L}_n := \tilde{K}_n / [\text{deck}(\tilde{K}_n), \text{deck}(\tilde{K}_n)],$$

which is an abelian covering (\tilde{L}_n, p_n, K) of K (cf. e.g. [29], p. 423, or [35], Theorem 2.2.10) such that $\text{deck}(\tilde{L}_n) = \Gamma_n$. The latter is a free abelian group with as many generators as the number of lacunas ℓ_σ , $|\sigma| \leq n-1$. Let us mention that the abelian coverings \tilde{L}_n , as well as their non-abelian counterparts \tilde{K}_n , are fractafolds in the sense of Strichartz [31] (see also [33, 34]). See figure 4 for a portion of \tilde{L}_2 , which is an example of a fundamental domain in the sense of Proposition A.4. Notice that x_i and x'_i project to the same point on K .

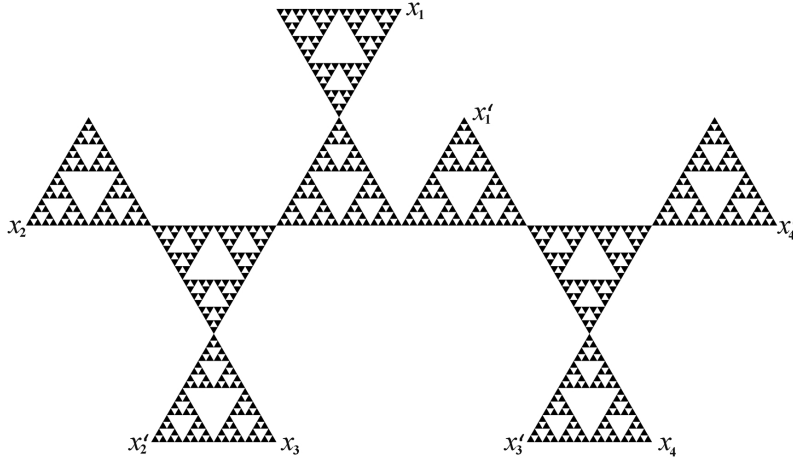


FIGURE 4. A fundamental domain for \tilde{L}_2 .

Lemma 3.2. *The above constructed potential f_ω of an n -exact topological 1-form ω is a $\text{deck}(\tilde{K}_n)$ -affine function on the covering space \tilde{K}_n , hence it can be considered as a Γ_n -affine function on the abelian covering space \tilde{L}_n .*

Proof. As already observed, the r.h.s. in (3.1) is clearly $\text{deck}(\tilde{K}_n)$ -invariant, namely

$$f(x) - f(x_0) = f(gx) - f(gx_0), \quad \forall g \in \text{deck}(\tilde{K}_n),$$

or, equivalently, $f(gx_0) - f(x_0) = f(gx) - f(x)$, namely the quantity $\varphi(g) = f(gx) - f(x)$ only depends on the group element g , and gives rise to a function on the group $\text{deck}(\tilde{K}_n)$, which is automatically continuous as this group is discrete. Moreover, for $g, h \in \text{deck}(\tilde{K}_n)$,

$$\varphi(gh) = f(ghx) - f(x) = (f(ghx) - f(hx)) + (f(hx) - f(x)) = \varphi(g) + \varphi(h),$$

that is φ is a homomorphism from $\text{deck}(\tilde{K}_n)$ to $(\mathbb{R}, +)$. \square

The family $\{(\tilde{L}_n, p_n, K) : n \in \mathbb{N}\}$ is projective too, and we denote by \tilde{L} the projective limit space. The projective limit Γ of the groups Γ_n is the direct product of countably many copies of \mathbb{Z} , where generators can be identified with lacunas, and coincides with the first Čech homology group (cf. e.g. [11], Theorem X.3.1, p. 261), which we shall denote by $\check{H}_1(K)$. The group $\tilde{\pi}_1(K)$ projects surjectively on $\check{H}_1(K)$.

Definition 3.3. We call *Uniform Universal Abelian Covering* of K the projective limit $\tilde{L} = \varprojlim \tilde{L}_n$, topologized by the projective limit topology.

We list below some properties of the spaces \tilde{K} and \tilde{L} that are needed in the sequel. We refer to [3] for other interesting properties.

Proposition 3.4.

- (i) \tilde{K} and \tilde{L} have the unique path-lifting property.
- (ii) \tilde{K} and \tilde{L} are path-wise connected.
- (iii) Γ is the direct product of countably many copies of \mathbb{Z} .

Proof. Property (i) for \tilde{K} follows by Corollary 74 in [3]. Property (ii) for \tilde{K} follows by Corollary 83 in [3]: indeed, since K is geodesic, it is uniformly (locally) path-wise connected (cf. Definition 66 in [3]). The corresponding properties of \tilde{L} follow, since \tilde{K} projects surjectively on \tilde{L} . Property (iii) follows by the definition of Γ and [3], Section 7. \square

Lemma 3.5. *For any Γ -affine function f on \tilde{L} there exists $n \in \mathbb{N}$ and a Γ_n -affine function f_n on \tilde{L}_n such that f_n lifts to f .*

Proof. This is the same as saying that the homomorphism φ associated with f satisfies $\varphi(g_\sigma) = 0$, for $|\sigma|$ large enough, where g_σ denotes the homotopy class of the lacuna ℓ_σ . Assume, by contradiction, that φ is continuous, and non-trivial on infinitely many elements $g_n = g_{\sigma_n}$. Recall that a sequence h_n in Γ converges to h in the projective limit topology iff, for any $k \in \mathbb{N}$, $q_k(h_n) = q_k(h)$ for sufficiently large n , where $q_k : \Gamma \rightarrow \Gamma_k$ is the projection; therefore, for any sequence $\{k_n\} \subset \mathbb{Z}$, $\lim_N \prod_{n=1}^N g_n^{k_n} = \prod_{n=1}^\infty g_n^{k_n}$ in the projective limit topology. As a consequence,

$$\varphi \left(\prod_{n=1}^\infty g_n^{k_n} \right) = \sum_{n=1}^\infty k_n \varphi(g_n).$$

However, one may always find a sequence of integers $\{k_n\}_{n \in \mathbb{N}}$ such that the series above diverges. \square

By the general theory of Dirichlet forms (see for example [13]), the space of *locally finite energy functions* $\tilde{\mathcal{F}}_{n,\text{loc}}$ on \tilde{K}_n is defined as those functions which coincide, on any open set of a suitable open cover of \tilde{K}_n , with a finite energy function in $\tilde{\mathcal{F}}_n$. Locally finite energy functions on \tilde{K}_n are, in particular, continuous. Potentials of locally exact smooth forms on K will be locally finite energy functions on the above considered covers.

Lemma 3.6. (i) A quadratic (energy) form $\mathcal{E}_\Gamma : \mathcal{A}(\Gamma, \tilde{L}) \rightarrow [0, +\infty]$ is well defined on the space $\mathcal{A}(\Gamma, \tilde{L})$ of Γ -affine functions on the covering space \tilde{L} by

$$(3.2) \quad \mathcal{E}_\Gamma[f] = \lim_n \left(\frac{5}{3} \right)^n \sum_{e \in E_n} |\partial f(e)|^2,$$

where the quantity $\partial f(e) := f(\tilde{e}_+) - f(\tilde{e}_-)$ does not depend on the choice of the lifting $\tilde{e} \subset \tilde{L}$ of $e \in E_*(K)$.

(ii) The energy of a Γ -affine function f is finite if and only if f is the potential of a locally exact form ω on K , and, in that case, $\mathcal{E}_\Gamma[f] = \|\omega\|_{\mathcal{H}}^2$. We shall write $df = \omega$.

Proof. (i) Let \tilde{e}^1, \tilde{e}^2 be two liftings, $\tilde{e}_n^1, \tilde{e}_n^2$ the corresponding projections on \tilde{L}_n , $g_n \in \Gamma_n$ be such that $g_n(\tilde{e}_n^1) = \tilde{e}_n^2$. The family $\{g_n\}$ is a projective sequence of deck transformations, which defines a deck transformation g on \tilde{L} satisfying $g(\tilde{e}^1) = \tilde{e}^2$. Since f is Γ -affine its variation is the same for all liftings. Since f is the lifting of a continuous function on \tilde{L}_m for some m , the sequence above is increasing for $n > m$, and this shows the second statement.

(ii) If f is a Γ -affine function of finite energy then, by Lemma 3.5, f is the lifting of a Γ_n -affine function f_n on \tilde{L}_n . Set $f_\sigma = f_n|_{C_\sigma}$ for $|\sigma| = n$. Since the covering projection from \tilde{L}_n to K is one to one on cells of level n , we get the desired form by glueing the df_σ 's. Conversely, the existence of a potential of a locally exact form has been already shown above, and the equality $\mathcal{E}_\Gamma[f] = \|\omega\|_2^2$ follows by Lemma 2.7. \square

Notice that the quadratic form just defined on Γ -affine functions on the covering space \tilde{L} reduces to the standard Dirichlet form on the gasket K when evaluated on periodic functions, i.e. on (liftings of) functions on K . This is also the reason why the notation $df = \omega$ is consistent with the usual notation for the derivation of a finite energy function on K .

By Proposition 2.12 (ii), any locally exact topological form modulo exact topological forms may be uniquely written as a finite linear combination of the dz_σ , the same result holding for locally exact smooth forms modulo exact smooth forms. Therefore, denoting by $B^1C(K)$ the space of exact topological 1-forms, the following definition makes sense.

Definition 3.7. We define $B^1(K, \mathbb{R})$ as the space of exact forms on K , and

$$(3.3) \quad H_{dR}^1(K, \mathbb{R}) = \frac{\Omega_{\text{loc}}^1 C(K)}{B^1C(K)} = \frac{\Omega_{\text{loc}}^1(K)}{B^1(K)}$$

as the *algebraic de Rham cohomology group* for the Sierpinski gasket.

Remark 3.8. Since the group $\Gamma = \check{H}_1(K)$ has no torsion, its homological information is fully recovered by the group $\check{H}_1(K, \mathbb{R}) = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem 3.9 (de Rham cohomology theorem). *There is a one to one correspondence between locally exact topological forms and their potentials (up to additive constants) such that $\int_\gamma \omega = f_\omega(x_1) - f_\omega(x_0)$ for any path $\gamma \subset K$, where x_0, x_1 are the end-points of a lifting of γ to \tilde{L} . When the locally exact form is smooth the corresponding potential has finite energy. Any*

class in $H_{dR}^1(K, \mathbb{R})$ has a smooth representative. The pairing $\langle \gamma, \omega \rangle = \int_\gamma \omega$ between continuous paths and locally exact forms gives rise to a nondegenerate pairing between elements of the group $\check{H}_1(K, \mathbb{R})$ and elements of $H_{dR}^1(K, \mathbb{R})$. Such a pairing is indeed a duality.

Proof. The first and second statements follow by the Lemmas above. The third follows by eq. (3.3). As for the last statement, observe that, for any continuous closed path γ in K and for any n , we may associate with γ its singular homology class $[\gamma]_n \in H_1(T_n)$, and then the projective limit $[\gamma] = \varprojlim [\gamma]_n \in \Gamma = \varprojlim H_1(T_n)$. If ω is k -exact, and φ_ω the associated homomorphism, then

$$\varphi_\omega([\gamma]) = \langle \varprojlim [\gamma]_n, \omega \rangle = \langle [\gamma]_k, \omega \rangle = \int_\gamma \omega.$$

Since the pairing above is trivial when the form is exact, we get a pairing $\Gamma \times H_{dR}^1(K, \mathbb{R}) \rightarrow \mathbb{R}$. Such pairing clearly extends to a pairing $\check{H}_1(K, \mathbb{R}) \times H_{dR}^1(K, \mathbb{R}) \rightarrow \mathbb{R}$.

Now we prove the duality relation. On the one hand, $H_{dR}^1(K, \mathbb{R})$ is isomorphic to $\varinjlim H_{dR}^1(T_n, \mathbb{R})$, topologized with the direct limit topology. On the other hand $\check{H}_1(K, \mathbb{R}) = \varprojlim H_1(T_n, \mathbb{R})$, topologized with the projective limit topology. The thesis follows by the classical duality result for T_n . \square

3.2. A metric on the Uniform Universal Abelian Covering \tilde{L} . In Section 2.6 we have seen that the introduction of the norm N on sequences selects both the space \mathcal{H}_N of 1-forms and the class of paths with finite effective length in such a way that the corresponding integral exists and is finite. The notion of path with finite effective length may also be read on the Uniform Universal Abelian Covering \tilde{L} , where the norm on sequences induces a (possibly infinite) distance d_N , hence splits the space in d_N -components. A path has finite effective length *iff* its lifting to \tilde{L} is contained in a single d_N -component. Forms with finite $\|\cdot\|_N$ norm have a finite, continuous potential on any d_N -component of \tilde{L} .

Clearly, by replacing the norm N with another norm on sequences, we may enlarge the class of 1-forms which may be lifted to (exact) 1-forms on (any d_N -component of) \tilde{L} , the key property for which the construction works being the connectedness of K by paths with finite effective length. This property is not satisfied in an extreme way when $\|\cdot\|_N$ coincides with the norm on \mathcal{H} . Indeed, this choice will restrict the d_N -components in such a way that their projection to K does not contain any edge, that is to say the potentials of such forms are defined in an extremely small space. Equivalently, no edge has finite effective length, cf. Remark 3.16.

We make use here of the norms N and N' on sequences $\{a \in \mathbb{R}^\Sigma\}$ introduced in Section 2.6. The metric d_N considered in the following will take also the value $+\infty$, therefore it splits the space in d_N -components, namely maximal subsets of points with mutually finite distance. Denoting by z_σ the Γ_n -affine potential on \tilde{L}_n of the n -exact form dz_σ , $n = |\sigma| + 1$, and by $\varphi_\sigma : \Gamma \rightarrow \mathbb{R}$ the corresponding homomorphism, we consider the function

$$(3.4) \quad d_N(x, y) = N'(z_\bullet(x) - z_\bullet(y)).$$

Lemma 3.10. *The function d_N is a Γ -invariant metric which is finer than the projective limit topology. If γ is a path in K and $\tilde{\gamma}$ is a lifting on \tilde{L} , the effective length of γ may be equivalently defined as $\lambda(\gamma) := d_N(\tilde{\gamma}(1), \tilde{\gamma}(0))$.*

Proof. The value $d_N(x, y)$ is obtained by composing the norm N' on sequences indexed by Σ with the (semi-definite) distances $d_\sigma(x, y) = |z_\sigma(y) - z_\sigma(x)|$. Therefore, on the one hand d_N

is a (possibly semi-definite) metric on \tilde{L} , on the other hand the topology induced by d_N is stronger than the weak topology induced by the z_σ 's, which is the projective limit topology, by Lemma A.4 in the Appendix. Since the projective limit topology is Hausdorff, this shows at once that d_N is positive definite and that is finer than the projective limit topology. Finally, we have $d_N(gx, gy) = N'(z_\bullet(gy) - z_\bullet(gx)) = N'(z_\bullet(y) - z_\bullet(x)) = d_N(x, y)$ for all $g \in \Gamma$. The last statement follows by the given definitions. \square

Lemma 3.11. *Let x be a point in \tilde{L} , $g \in \Gamma$. Then, the quantity $\ell_N(g) := d_N(x, gx)$ does not depend on x , and $\ell_N(g) = 0$ iff g is the identity. The set $\Gamma_N = \{g \in \Gamma : d_N(x, gx) < \infty\}$ does not depend on x , and is a subgroup of Γ . The function $\ell_N(g)$ is a length function on Γ_N .*

Proof. For any $\sigma \in \Sigma$, let $\varphi_\sigma \in \text{hom}(\Gamma, \mathbb{R})$ be the homomorphism associated to the Γ -affine function z_σ on \tilde{L} in such a way that $z_\sigma(gx) - z_\sigma(x) = \varphi_\sigma(g)$ for all $g \in \Gamma$. Let us denote by $\varphi_\bullet(g) \in \mathbb{R}^\Sigma$ the sequence $\sigma \mapsto \varphi_\sigma(g)$. Equation (3.4) then shows that

$$d_N(x, gx) = N(z_\bullet(gx) - z_\bullet(x)) = N(\varphi_\bullet(g)),$$

and the first statement follows. Since d_N is a pseudo-metric, $\ell_N(g) = 0$ means $gx = x$ for any x , namely $g = e$. The last two properties are obvious. \square

Lemma 3.12. *The projection map p restricted to a d_N -component is surjective \iff for all $x, y \in K$ there is a continuous path γ in K between them which has finite effective length.*

Proof. (\Leftarrow) Let us fix $\tilde{x}_0 \in \tilde{L}$, and let $x_0 := p(\tilde{x}_0)$. Then, for any $x \in K$ there is a continuous path γ in K , starting in x_0 and ending in x , which has finite effective length. Denote by $\tilde{\gamma}$ its unique lifting to a path in \tilde{L} starting at $\tilde{x}_0 \in \tilde{L}$. Then $\pi(\tilde{\gamma}(1)) = x$, and $\tilde{\gamma}(1)$ belongs to the same d_N -component of \tilde{x}_0 .

(\Rightarrow) Let $x, y \in K$. By assumption, there are $\tilde{x}, \tilde{y} \in \tilde{L}$ such that $d_N(\tilde{x}, \tilde{y}) < \infty$ and $p(\tilde{x}) = x$, $p(\tilde{y}) = y$. Because of Proposition 3.4 (ii), there is a continuous path $\tilde{\gamma}$ in \tilde{L} between \tilde{x} and \tilde{y} . Set $\gamma := p \circ \tilde{\gamma}$, which automatically has finite effective length. \square

Lemma 3.13. *Elementary paths have finite effective length. Any d_N -component of \tilde{L} projects surjectively on K .*

Proof. By Lemma 2.30, edges have finite effective length. Now we observe that the effective length is sub-additive. Indeed, if γ_1, γ_2 are consecutive paths, $\tilde{\gamma}_1$ is a lifting of γ_1 starting from some point $\tilde{x}_0 \in \tilde{L}$, and $\tilde{\gamma}_2$ is a lifting of γ_2 starting from $x := \tilde{\gamma}_1(1)$, then

$$\lambda(\gamma_1 \cdot \gamma_2) = d_N(\tilde{\gamma}_1(0), \tilde{\gamma}_2(1)) \leq d_N(\tilde{\gamma}_1(0), x) + d_N(x, \tilde{\gamma}_2(1)) = \lambda(\gamma_1) + \lambda(\gamma_2).$$

The first statement follows. As for the second, the thesis is equivalent to the connectedness of K by means of paths of finite effective length, as shown in Lemma 3.12. We have shown in Lemma 2.22 that a vertex $v_0 \in V_0$ can be connected to any vertex of level p by an elementary path consisting of at most 1 edge of level j for any $j \leq p$, thus proving that v_0 can indeed be connected to any point x in K by a path consisting of (possibly infinitely many) edges, at most 1 of them for any level. The thesis follows by the estimate in Lemma 2.30 and sub-additivity. \square

3.3. Potentials of smooth 1-forms. Indeed the results are formulated for the elements of the closure of $\Omega^1(K)$ in \mathcal{H} w.r.t. the norm $\|\cdot\|_N$, namely for elements of \mathcal{H}_N .

Lemma 3.14. *Let $\omega = dU_E + \sum_\sigma k_\sigma dz_\sigma \in \mathcal{H}_N$. For any d_N -component $\tilde{L}_0 \subset \tilde{L}$, we may associate to ω a function $U = U_E + U_H$, where U_E was described in Proposition 2.23 and,*

$\forall x_0 \in \tilde{L}_0$, U_H may be written as

$$U_H(x) = \sum_{\sigma} k_{\sigma}(z_{\sigma}(x) - z_{\sigma}(x_0)).$$

The series defining U_H converges uniformly on compact sets, and U_H is a d_N -continuous Γ_N -affine function on \tilde{L}_0 . In particular, U is a potential for ω , namely, for any continuous path γ in K , $\lambda(\gamma) < \infty$, and any lifting $\tilde{\gamma}$ of γ to \tilde{L} , it holds $\int_{\gamma} \omega = U(\tilde{\gamma}(1)) - U(\tilde{\gamma}(0)) < \infty$.

Proof. Given two points $x_1, x_2 \in \tilde{L}_0$, we have

$$|U_H(x_2) - U_H(x_1)| = \left| \sum_{\sigma} k_{\sigma}(z_{\sigma}(x_2) - z_{\sigma}(x_1)) \right| \leq N'(z_{\bullet}(x_2) - z_{\bullet}(x_1)) N(k_{\bullet}) \leq d_N(x_1, x_2) \|\omega\|_N.$$

As a consequence, U_H is Lipschitz d_N -continuous. In particular, if $\ell(g) < \infty$, then x and gx belong to the same d_N -component, and $U_H(gx) - U_H(x) = \sum_{\sigma} k_{\sigma} \varphi_{\sigma}(g)$, namely U_H is Γ_N -affine. Since U_E is continuous on K , it lifts to a Γ -invariant function on \tilde{L} , continuous in the projective limit topology, hence also in the (stronger) d_N -topology. The last statement easily follows. \square

Theorem 3.15. (i) Any form in \mathcal{H}_N has a Γ_N -affine potential on any d_N -component of \tilde{L} ;
(ii) the integral of a form in \mathcal{H}_N along a path γ with finite effective length coincides with the variation of the potential at the end points of a lifting of γ ;
(iii) such integral gives a nondegenerate pairing between Γ_N and $\mathcal{H}_N/B^1(K)$. Indeed, the space $\mathcal{H}_N/B^1(K)$ is the Banach space dual of $\Gamma_N \otimes_{\mathbb{Z}} \mathbb{R}$.

Proof. The first two statements have been proved above. As for the third, we observe that, for $g \in \Gamma_N$, $\omega \in \mathcal{H}_N$, the pairing $\langle g, \omega \rangle$ may be defined as $\int_{\gamma} \omega$, where γ is any closed path giving a representative of g . Then, $g \in \Gamma_N$ iff the sequence $\{\langle g, dz_{\sigma} \rangle\}_{\sigma \in \Sigma}$ belongs to $\ell_{N'}(\Sigma)$. On the other hand, we proved in Theorem 3.9 that $\tilde{H}_1(K, \mathbb{R}) = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ may be identified with the dual of the space of cohomology classes of locally exact 1-forms. Since any such class may be uniquely described through its harmonic representative $\omega = \sum_{\sigma} k_{\sigma} dz_{\sigma}$, namely by the eventually zero sequence $k := \{k_{\sigma}\}$, the elements of $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ may be identified with infinite sequences $\alpha = \{\alpha_{\sigma}\}$, with $\langle \alpha, \omega \rangle = \sum_{\sigma} \alpha_{\sigma} k_{\sigma}$. Then, the sequence $\alpha = \{\alpha_{\sigma}\}$ belongs to $\Gamma_N \otimes_{\mathbb{Z}} \mathbb{R}$ iff $\{\alpha_{\sigma}\}_{\sigma \in \Sigma} = \{\langle \alpha, dz_{\sigma} \rangle\}_{\sigma \in \Sigma} \in \ell_{N'}(\Sigma)$, or, equivalently, $\Gamma_N \otimes_{\mathbb{Z}} \mathbb{R} \cong \ell_{N'}(\Sigma)$. Since $\mathcal{H}_N/B^1(K)$ may be identified with $\ell_N(\Sigma)$, the thesis follows. \square

We observe that, for any choice of the norm N such that the d_N -components of \tilde{L} projects surjectively on K , it is possible to construct potentials of any form in \mathcal{H}_N on the d_N -components. The choice of the norm N affects both the size of the d_N -components and the smoothness of the forms that can be lifted there. In particular, we may lift less regular 1-forms but on smaller d_N -components. However, lifting all forms in the tangent bimodule \mathcal{H} means getting d_N -components as small as to contain no edge.

Proposition 3.16. Choosing N such that $\|\cdot\|_N = \|\cdot\|_{\mathcal{H}}$, no edge has finite effective length.

Proof. We prove the statement for the edge e_1 opposite to the vertex $p_1 \in V_0$, the other cases follow in a similar way. The norms $\|\cdot\|_N$ and $\|\cdot\|_{\mathcal{H}}$ coincide if we set $N(a)^2 = 5/6 \sum_{\sigma} (5/3)^{|\sigma|} |a_{\sigma}|^2$. Therefore $\lambda(e_1)^2 = 6/5 \sum_{\sigma} (3/5)^{|\sigma|} |\int_{e_1} dz_{\sigma}|^2$. Since $\int_{e_1} dz_{\sigma} = -1/3$ if the multi-index σ does not contain the index 1, and vanishes otherwise, we get $\lambda(e_1)^2 = (2/15) \sum_n (6/5)^n = +\infty$. \square

APPENDIX A. THE PROJECTIVE LIMIT TOPOLOGY ON \tilde{L} IS GENERATED BY POTENTIALS

Lemma A.1. *Let $C_{\sigma i}$ be one of the three subcells of the cell C_σ , denote by z_σ^i the potential of dz_σ on $C_{\sigma i}$, and by $x_\sigma^i = w_\sigma(p_i)$ the common vertex of $C_{\sigma i}$ and C_σ . Then*

- (a) *The set $\{x \in C_{\sigma i} : z_\sigma^i(x) = z_\sigma^i(x_\sigma^i)\}$ coincides with the intersection A_σ^i of $C_{\sigma i}$ with the axis of the edge $e_{\sigma i}^i = w_{\sigma i}(e_i)$ opposite to x_σ^i in $C_{\sigma i}$.*
- (b) *All points in A_σ^i are vertices.*

Proof. It is not restrictive to assume that $\sigma = \emptyset$, $i = 1$, $z(p_1) := z_\emptyset^1(p_1) = 0$. We first prove the following statement.

Claim A.2. *For any $n \in \mathbb{N}$, denote by $\mathbf{1}_n$ the multi-index of length n and taking only the value 1, and let $\Theta_n := \{\mathbf{1}_k : k = 1, \dots, n\}$. Then,*

$$(A.1) \quad C_1 = C_{\mathbf{1}_n} \cup \bigcup_{\rho \in \Theta_{n-1}} C_{\rho 0} \cup C_{\rho 2}.$$

- (i) *If $x \in C_{\rho 0}$, $\rho \in \Theta_{n-1}$, and $z(x) = 0$ then $x = w_{\rho 0}(p_2)$, hence is on the axis $A := A_\emptyset^1$. Analogously, if $x \in C_{\rho 2}$, $\rho \in \Theta_{n-1}$, and $z(x) = 0$ then $x = w_{\rho 2}(p_0) \in A$.*
- (ii) *The values of z at the points $w_{\mathbf{1}_n}(p_0)$, $w_{\mathbf{1}_n}(p_2)$ are, respectively, $-1/6 \cdot 5^{-n+1}$, $1/6 \cdot 5^{-n+1}$.*

Proof of the Claim. The statement clearly holds for $n = 1$. Suppose now it is true for some n . Since $C_{\mathbf{1}_n} = C_{\mathbf{1}_{n-1}0} \cup C_{\mathbf{1}_{n-1}1} \cup C_{\mathbf{1}_{n-1}2}$, equality (A.1) still holds. By harmonic extension, the boundary values of z on $C_{\mathbf{1}_{n-1}0}$ are $-1/6 \cdot 5^{-n+1}$, $-1/6 \cdot 5^{-n}$ and 0, hence, by the maximum principle, the value 0 is assumed only on the vertex, proving (i). The proof of (ii) also follows by harmonic extension. \square

Now we turn to the proof of the Lemma. If $z(x) = 0$, either $x \in A$ or $x \in \cap_n C_{\mathbf{1}_n}$, which means $x = p_1 \in A$. Conversely, if $x \in A$, either x is a vertex and $z(x) = 0$ or $x \in \cap_n C_{\mathbf{1}_n}$, which means $x = p_1$ hence $z(x) = 0$. Both (a) and (b) then follow. \square

Lemma A.3. *For any $g \in \Gamma_n$, there exists $|\sigma| < n$ such that $\varphi_\sigma(g)$ is a non-vanishing integer, where φ_σ is the homomorphism associated with the Γ_n -affine potential z_σ .*

Proof. The element g may be uniquely decomposed as $g = \prod_{|\tau| < n} g_\tau^{k_\tau}$, where g_τ denotes the homology class of the lacuna ℓ_τ according to the identification $\Gamma_n = H_1(T_n)$. If we choose σ of minimal length such that $k_\sigma \neq 0$, we have

$$\varphi_\sigma(g) = \sum_{|\sigma| \leq |\tau| < n} k_\tau \varphi_\sigma(g_\tau) = \sum_{|\sigma| \leq |\tau| < n} k_\tau \int_{\ell_\tau} dz_\sigma = k_\sigma,$$

where we used the fact that, as observed at the beginning of Subsection 2.4, $\int_{\ell_\tau} dz_\sigma$ is non-zero only if $\tau \leq \sigma$. \square

Proposition A.4. *The weak topology $\mathcal{T}(z_\sigma)$ induced by $\{z_\sigma : \sigma \in \Sigma\}$ on \tilde{L} coincides with the projective limit topology.*

Proof. We shall prove that, given a point $\tilde{x} \in \tilde{L}$ and one of its neighborhoods \tilde{U} in the projective limit topology, there exists a set Ω , open in the weak topology induced by $\{z_\sigma : \sigma \in \Sigma\}$, such that $x \in \Omega \subseteq \tilde{U}$. This proof will in some points split in three cases:

- (c1) $p(\tilde{x}) \notin V_*$,
- (c2) $p(\tilde{x}) \in V_0$,
- (c3) $p(\tilde{x}) \in V_* \setminus V_0$,

where $p : \tilde{L} \rightarrow K$ is the covering projection. In the course of the proof, we will use the standard notation X° , resp. ∂X for the (topological) interior, resp. boundary, of $X \subset K$. To avoid misunderstanding, we will denote by C_σ° , resp. bC_σ , the combinatorial interior, resp. boundary, of a cell C_σ . Observe that $C_\sigma^\circ = C_\sigma^\circ$ and $\partial C_\sigma = bC_\sigma \iff C_\sigma$ doesn't contain one of the vertices p_0, p_1, p_2 .

About the neighborhood \tilde{U} . By definition, there exists $n \in \mathbb{N}$ such that \tilde{U} is the preimage in \tilde{L} of a neighborhood U of $x_0 \in \tilde{L}_n$, where \tilde{x} projects onto x_0 . It is not restrictive to assume, possibly passing to a higher covering, that

(c1-c2) the open set U is the interior of a cell of level n in \tilde{L}_n .

(c3) the open set U is a butterfly shaped neighborhood made of two cells of level n in \tilde{L}_n in such a way that $p_n(U)$ is not contained in a cell of level $n-1$, where $p_n : \tilde{L}_n \rightarrow K$ is the covering projection.

The choice of a fundamental domain. As a closed fundamental domain \mathcal{F} in \tilde{L}_n , we pick a finite union of closed cells of level n in \tilde{L}_n such that \mathcal{F} is connected, $p_n(\mathcal{F}) = K$ and $p_n|_{\mathcal{F}^\circ}$ is injective, and with the further property that, for any $|\tau| = n-1$, $p_n^{-1}(C_\tau) \cap \mathcal{F}$ is connected. We also require that

(c1-c2) the neighboring cells of U in \tilde{L}_n , whose projection to K lie in the same cell of level $n-1$ containing $p_n(U)$, still belong to \mathcal{F} . If $p_n(U) = C_{\sigma i}^\circ$ [i.e. $p_n(U) = C_{\sigma i}^\circ$ or $p_n(U) = C_{\sigma i}^\circ \cup \{p_n(x_0)\}$], we get in particular that U is in the middle of the preimage $p_n^{-1}(C_\sigma) \cap \mathcal{F}$.

(c3) same as above for the two subcells of the butterfly neighborhood U . If $p_n(U) = (C_{\sigma i} \cup C_{\rho j})^\circ$ [where, by the above assumption, $\sigma \neq \rho$ and $i \neq j$], we get in particular that $U \cap p_n^{-1}(C_\sigma)$ is in the middle of the preimage $p_n^{-1}(C_\sigma) \cap \mathcal{F}$, and $U \cap p_n^{-1}(C_\rho)$ is in the middle of the preimage $p_n^{-1}(C_\rho) \cap \mathcal{F}$.

The normalization of the z_τ 's. We have asked the preimage in \mathcal{F} of any cell C_τ , $|\tau| = n-1$ to be connected. Since such preimage consists of three cells of level n , only one of them is intermediate, namely has a vertex in common with the others. For $|\tau| = n-1$, we set z_τ to be zero on the third vertex of such intermediate cell, so that the range of z_τ on $p_n^{-1}(C_\tau) \cap \mathcal{F}$ is $[-1/2, 1/2]$. We normalize the z_τ for $|\tau| \leq n-2$ such that, again, the range of z_τ on $p_n^{-1}(C_\tau) \cap \mathcal{F}$ is $[-1/2, 1/2]$. In particular,

(c1) the range of z_σ on $U = p_n^{-1}(C_{\sigma i}^\circ) \cap \mathcal{F}$ is $(-1/6, 1/6)$, because U is the intermediate cell, so that, by Lemma A.1, $z_\sigma(x_0) \neq 0$,

(c2) the range of z_σ on $U = p_n^{-1}(C_{\sigma i}^\circ \cup \{p_n(x_0)\}) \cap \mathcal{F}$ is $(-1/6, 1/6)$ and, by Lemma A.1, $z_\sigma(x_0) = 0$,

(c3) the ranges of z_σ and z_ρ on $U = p_n^{-1}((C_{\sigma i} \cup C_{\rho j})^\circ) \cap \mathcal{F}$ are equal to $(-1/6, 1/6)$ and, by Lemma A.1, $z_\sigma(x_0) = z_\rho(x_0) = 0$.

\mathcal{F}° is open in the topology $\mathcal{T}(z_\sigma)$. By definition of \mathcal{F} , for any $x \in \mathcal{F}^\circ$, and for any $|\tau| < n$, the position $z_\tau^\mathcal{F}(p_n(x)) := z_\tau(x)$ gives a well defined function on $p_n(\mathcal{F}^\circ)$. As a consequence, with the normalization above, $z_\tau^\mathcal{F}$ takes values in $(-1/2, 1/2)$ on the open cell C_τ° , and is constant on the other cells, with values $-1/3, 0, 1/3$. Therefore, for any $|\tau| < n$, $\{z_\tau(x) : x \in \mathcal{F}^\circ\} = (-1/2, 1/2)$. If $x \notin \mathcal{F}$, there exists $x' \in \mathcal{F}$ and a non trivial $g \in \Gamma_n$ such that $x = gx'$. By Lemma A.3 there exists $|\tau| < n$ such that $\varphi_\tau(g)$ is a non zero integer, hence $z_\tau(x) = \varphi_\tau(g) + z_\tau(x') \in (-\infty, -1/2] \cup [1/2, +\infty)$. Also, if $x \in \partial\mathcal{F}$, $p_n(x) \in V_n \setminus V_0$, hence $\exists! \tau$, $|\tau| < n$ such that $p_n(x)$ is a vertex of ℓ_τ and $z_\tau(x) = \pm 1/2$. Then,

$$(A.2) \quad \{x \in \tilde{L}_n : z_\tau(x) \in (-1/2, 1/2), |\tau| < n\} = \mathcal{F}^\circ$$

The construction of Ω .

- (c1) Set $\Omega = \bigcap_{\tau \neq \sigma, |\tau|=n-1} z_\tau^{-1}(-1/2, 1/2) \cap z_\sigma^{-1}\{(-1/6, 0) \cup (0, 1/6)\}$. The result above implies $\Omega \subset \mathcal{F}^\circ$. By the chosen normalization, the values of $z_\sigma^\mathcal{F}$ on the cells different from C_σ can only be $-1/3$, 0 or $1/3$, hence the values in $(-1/6, 0) \cup (0, 1/6)$ are only assumed in $C_{\sigma i}^\circ \equiv C_{\sigma i}^\iota$. Therefore $\Omega \subset U$.
- (c2) Set $\Omega = \bigcap_{|\tau|=n-1} z_\tau^{-1}(-1/6, 1/6)$. Again $\Omega \subset \mathcal{F}^\circ$. Since $p_n(x_0)$ is in V_0 , the values of $z_\sigma^\mathcal{F}$ on the cells different from C_σ can only be $-1/3$ or $1/3$, namely the values $(-1/6, 1/6)$ are only assumed in $C_{\sigma i}^\circ \equiv C_{\sigma i}^\iota \cup \{p_n(x_0)\}$. Therefore $\Omega \subset U$.
- (c3) Set $\Omega = \bigcap_{|\tau|=n-1} z_\tau^{-1}(-1/6, 1/6)$. Again $\Omega \subset \mathcal{F}^\circ$. By construction, the removal of the cell C_σ° disconnects $p_n(\mathcal{F}^\circ)$, and we call $D_\sigma(-1/3), D_\sigma(0), D_\sigma(1/3)$ the (connected) components according to the value of $z_\sigma^\mathcal{F}$ on them. In the same way, the removal of the cell C_ρ° disconnects $p_n(\mathcal{F}^\circ)$, and we call $D_\rho(-1/3), D_\rho(0), D_\rho(1/3)$ the components according to the value of $z_\rho^\mathcal{F}$ on them. Note that, by the simple connectedness of T_n , $D_\sigma(0) = \{p_n(x_0)\} \sqcup C_\rho^\circ \sqcup D_\rho(-1/3) \sqcup D_\rho(1/3)$ and $D_\rho(0) = \{p_n(x_0)\} \sqcup C_\sigma^\circ \sqcup D_\sigma(-1/3) \sqcup D_\sigma(1/3)$, where \sqcup denotes disjoint union. Then, the prescription $z_\sigma^\mathcal{F}(y) \in (-1/6, 1/6)$ selects $C_{\sigma i}^\circ \cup D_\sigma(0)$, the prescription $z_\rho^\mathcal{F}(y) \in (-1/6, 1/6)$ selects $C_{\rho j}^\circ \cup D_\rho(0)$, both select $(C_{\sigma i} \cup C_{\rho j})^\circ$, implying $\Omega \subset U$.

Since in all three cases $\Omega \in \mathcal{T}(z_\sigma)$, we have proved the thesis. \square

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